MTH 210, Spring 2016
HW5 Solutions

Problem 1: Assume that

$$A\vec{x} = \vec{b} \quad \& \quad A\vec{y} = \vec{b}.$$

Then for all numbers t we have

$$\begin{aligned} &A(t\vec{x} + (1-t)\vec{y}) \\ &= tA\vec{x} + (1-t)A\vec{y} \\ &= t\vec{b} + (1-t)\vec{b} \\ &= [\cancel{t} + (1-\cancel{t})]\vec{b} \\ &= 1\vec{b} = \vec{b}. \quad /// \end{aligned}$$

Interpretation: The solutions to a system of linear equations form a "flat" shape, i.e., a "d-plane" for some dimension d.

Problem 2: Consider the matrix

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 0 & -1 \\ 3 & 1 & 2 \end{pmatrix}.$$

By repeatedly using the rule

$$(\text{2nd col. of } AA^n) = A(\text{2nd col. of } A^n)$$

we obtain ...

(a) $(\text{2nd col. of } AA) = A(\text{2nd col. of } A)$

$$= \begin{pmatrix} 1 & -1 & 2 \\ 2 & 0 & -1 \\ 3 & 1 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1+0+2 \\ -2+0-1 \\ -3+0+2 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix}$$

(b) $(\text{2nd col. of } AA^2) = A(\text{2nd col. of } A^2)$

$$= \begin{pmatrix} 1 & -1 & 2 \\ 2 & 0 & -1 \\ 3 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix} = \begin{pmatrix} 1+3-2 \\ 2+0+1 \\ 3-3-2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}$$

(c) (2nd col. of AA^3) = A (2nd col. of A^3)

$$= \begin{pmatrix} 1 & -1 & 2 \\ 2 & 0 & -1 \\ 3 & 1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 2-3-4 \\ 4+0+2 \\ 6+3-4 \end{pmatrix} = \begin{pmatrix} -5 \\ 6 \\ 5 \end{pmatrix}.$$

[My computer tells me that the full matrix is

$$A^4 = \begin{pmatrix} 101 & -5 & 100 \\ 20 & 6 & 5 \\ 155 & 5 & 156 \end{pmatrix},$$

which would have taken a lot longer to compute by hand. Good thing we didn't need to.]

Problem 3: Let $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

(a) $R_{30^\circ} = \begin{pmatrix} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ \end{pmatrix}$

$$= \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{pmatrix}.$$

$$R_{45^\circ} = \begin{pmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

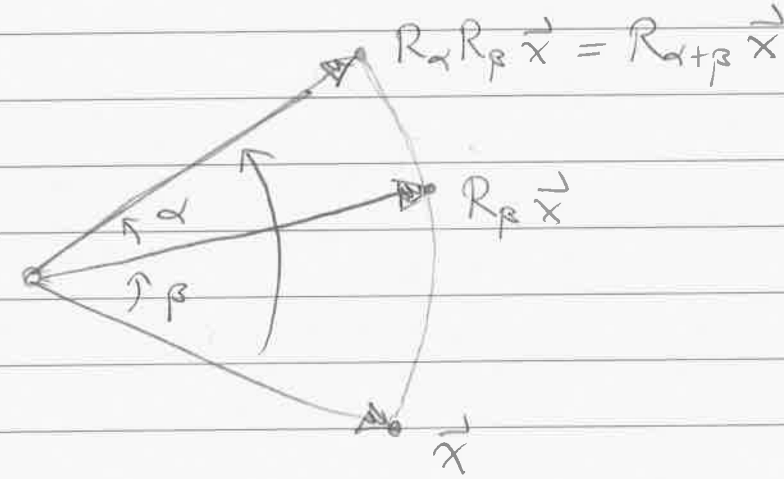
$$R_{60^\circ} = \begin{pmatrix} \cos 60^\circ & -\sin 60^\circ \\ \sin 60^\circ & \cos 60^\circ \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}.$$

$$R_{90^\circ} = \begin{pmatrix} \cos 90^\circ & -\sin 90^\circ \\ \sin 90^\circ & \cos 90^\circ \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

(b) I claim that for all angles α & β we have $R_\alpha R_\beta = R_{\alpha+\beta}$.

Indeed, the matrix on the right rotates each vector by angle $\alpha+\beta$ counterclockwise. The matrix on the left <u>first</u> rotates by angle β counterclockwise and <u>then</u> rotates by angle α counterclockwise. The result on either side is the same.

Picture:



[For example, we should have $R_{30°}R_{30°} = R_{60°}$.
Let's check that it works:

$$\frac{1}{2}\begin{pmatrix}\sqrt{3} & -1\\ 1 & \sqrt{3}\end{pmatrix}\frac{1}{2}\begin{pmatrix}\sqrt{3} & -1\\ 1 & \sqrt{3}\end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix}\sqrt{3} & -1\\ 1 & \sqrt{3}\end{pmatrix}\begin{pmatrix}\sqrt{3} & -1\\ 1 & \sqrt{3}\end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix}3-1 & -\sqrt{3}-\sqrt{3}\\ \sqrt{3}+\sqrt{3} & -1+3\end{pmatrix} = \frac{1}{4}\begin{pmatrix}2 & -2\sqrt{3}\\ 2\sqrt{3} & 2\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}1 & -\sqrt{3}\\ \sqrt{3} & 1\end{pmatrix} \quad \checkmark$$
]

(c) Expanding both sides of $R_{\alpha+\beta} = R_\alpha R_\beta$ gives

$$\begin{pmatrix}\cos(\alpha+\beta) & -\sin(\alpha+\beta)\\ \sin(\alpha+\beta) & \cos(\alpha+\beta)\end{pmatrix} = \begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}\cos\beta & -\sin\beta\\ \sin\beta & \cos\beta\end{pmatrix}$$

$$= \begin{pmatrix}\cos\alpha\cos\beta - \sin\alpha\sin\beta & -\cos\alpha\sin\beta - \sin\alpha\cos\beta\\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & -\sin\alpha\sin\beta + \cos\alpha\cos\beta\end{pmatrix}$$

Since the matrices are equal, they must have the same entries. In other words, for all α & β we must have

$$\begin{cases} \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \end{cases}.$$

[Now you know <u>why</u> these trigonometric angle sum formulas are true.]

Problem 4: The word "projection" is not important here. We might as well just call them "special" matrices.

(a) Suppose that P is special (i.e. suppose that $P^T = P$ & $P^2 = P$). Then the matrix $I-P$ is also special because

$$(I-P)^T = I^T - P^T = I - P \quad ✓$$

and

$$\begin{aligned}(I-P)^2 &= (I-P)(I-P) \\ &= II - IP - PI + PP \\ &= I - P - P + P^2 \\ &= I - P - \cancel{P} + \cancel{P} \\ &= I - P \quad ✓\end{aligned}$$

(b) If P is special then we have

$$P(I-P) = PI - PP = P - P^2 = P - P = 0.$$

[Remark: We'll learn next week what "projection" means. Then we'll see that, for example, if P is the projection onto a plane in $\mathbb{R}^3$ then $I-P$ is the projection onto the perpendicular line (and vice versa)]

Problem 5: Let A be a matrix and suppose that we have

$$A\vec{x}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad A\vec{x}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}, \quad A\vec{x}_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

for some vectors $\vec{x}_1, \vec{x}_2, \vec{x}_3$.

(a) Suppose that A has shape $m \times n$. Then the vectors must have shape $n \times 1$ and we must have $m = 3$ so that, e.g.,

$$A \vec{x}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}.$$

$$3 \times n \quad n \times 1 \qquad 3 \times 1$$

(b) Now define the $n \times 3$ matrix

$$X = \begin{pmatrix} | & | & | \\ \vec{x}_1 & \vec{x}_2 & \vec{x}_3 \\ | & | & | \end{pmatrix}$$

Using the formula

$$(j^{th} \text{ col of } AX) = A(j^{th} \text{ col. of } X)$$

from Problem 2 tells us that

$$AX = \begin{pmatrix} | & | & | \\ A\vec{x}_1 & A\vec{x}_2 & A\vec{x}_3 \\ | & | & | \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3 .$$

In other words, X is a "right inverse" of A.

[In one version of the homework I claimed that we must also have $n = 3$, but this is FALSE. Consider this example:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\underset{3\times 4}{A} \cdot \underset{4 \times 3}{X} = \underset{3\times 3}{I_3}.$$

In other words, it is possible for A to have a "right inverse" even when it <u>doesn't</u> have a "two sided inverse".

Problem 6: Now consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

We want to find vectors $\vec{x}_1, \vec{x}_2, \vec{x}_3$ such that

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \vec{x}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \vec{x}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \vec{x}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

We can solve the three systems separately to get

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 1 & -1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\Longrightarrow \vec{x}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix},$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

$$\Longrightarrow \vec{x}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix},$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$$\Longrightarrow \vec{x}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

and hence

$$X = \begin{pmatrix} \vec{x}_1 & \vec{x}_2 & \vec{x}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}.$$

Or we could solve the three systems simultaneously to get

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right).$$

It's up to you. In any case, we check that

$$AX = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1+0+0 & 0+0+0 & 0+0+0 \\ 1-1+0 & 0+1+0 & 0+0+0 \\ 1-1+0 & 0+1-1 & 0+0+1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3,$$

as desired.

This is not surprising because it was exactly the problem we were trying to solve.

Finally, let's investigate XA :

$$XA = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1+0+0 & 0+0+0 & 0+0+0 \\ -1+1+0 & 0+1+0 & 0+0+0 \\ 0-1+1 & 0-1+1 & 0+0+1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3 \, ,$$

Hey, that's cool. It turns out that X is also a "left inverse" of A, so we conclude that X is <u>the</u> inverse of A,

$$X = A^{-1} \, .$$