3/14/16

Welcome back.
HW5 : TBA.

Exam1 Statistics		HW4
Total	30	25
Average	22.2	23.5
Median	22.5	24
Terciles	21, 25	23, 25
St. Dev.	5.8	1.7

This is the point in a linear algebra class at which the teacher might launch into a discussion of the following abstract concepts:

- linear independence
- spanning set
- basis
- dimension
- vector space

Since we only have one semester together I'm not going to waste your time with such abstractions.

Instead, I think it is more appropriate to show you the two principal ways that linear algebra gets <u>used</u> in the real world:

- "least squares regression"
- "spectral analysis"

However, before moving on to applications, there is <u>one</u> concept from the abstract story that I want to tell you about (because it's the part of the story that's the most <u>useful</u>.)

Consider a linear system

$$A\vec{x} = \vec{b}$$

where A is an $m \times n$ matrix. I told you before that the solution of this system is a d-plane living in $\mathbb{R}^n$ and that we <u>probably</u> have

$$d = n - m$$

$$\underset{\text{solution}}{\text{dim. of the}} = \#\text{variables} - \#\text{equations}.$$

But I never told you what I meant by "probably". Today I will.

☆ Definition: Let A be an $m \times n$ matrix and consider the so-called "homogeneous" linear system

$$A\vec{x} = \vec{0}.$$

As I said, the solution of this solution probably has dimension $n-m$, but maybe not. In general we will write

$$\text{null}(A)$$

for the dimension of of the solution, and we will call this the <u>nullity</u> of the matrix A. Thus we "probably" have

$$\begin{aligned}\text{null}(A) &= n-m \\ &= \#\text{columns} - \#\text{rows}.\end{aligned}$$

///

Let's consider an example from HW3.

↓

Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 3 & 2 \end{pmatrix}.$$

We find the solution of the homogeneous system

$$A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

by using Gaussian elimination:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 1 & 0 \\ 2 & 3 & 2 & 0 \end{array}\right) \overset{\text{RREF}}{\rightsquigarrow} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{cases} x + 0 + z = 0 \\ \quad y + 0 = 0 \\ \qquad 0 = 0. \end{cases}$$

The pivot variables are x & y and the variable z is free. Letting $z = t$ gives the complete solution

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -t \\ 0 \\ t \end{pmatrix} = t \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix},$$

which is a line. Since a line is 1-dimensional we conclude that

$$\text{null}(A) = 1,$$

and this is unusual because we would expect $\text{null}(A) = 3 - 3 = 0$ for a typical 3×3 matrix.

We also discussed earlier that a deviation from $\text{null}(A) = n - m$ is related to the existence of "relationships" (usually called linear relations) among the columns of A.

In our example, the solution $(-1, 0, 1)$ corresponds to the linear relation

$$(-1)\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + 0 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

In general, we obtain a new linear relation for each free parameter in the solution, so we can also say the following:

null(A) = # essentially different linear relations among the columns of A.

I also mentioned that deviation from the expected solution is related the existence of linear relations among the <u>rows</u> of A, but this is more mysterious. The mystery can be phrased as follows.

☆ Fundamental Mystery of Linear Algebra:

<u>Why</u> is the existence of linear relations among the rows of a matrix be related to the existence of linear relations among the columns?

Shouldn't these two properties be independent of each other?

Example: Our matrix A has a row relation

$$\underbrace{(1\ 1\ 1)}_{\text{1st row}} + \underbrace{(1\ 2\ 1)}_{\text{2nd row}} = \underbrace{(2\ 3\ 2)}_{\text{3rd row}},$$

but it is not clear what this has to do with the column relation

$$\underbrace{\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}}_{\text{1st col}} = \underbrace{\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}}_{\text{3rd col}}.$$

The fact that these two concepts <u>are</u> related is a nontrivial and deep fact that I think you should know. To emphasize its importance I'll give it an impressive sounding name.

☆ Fundamental Theorem of Linear Algebra:

Let A be an $m \times n$ matrix with nullity $\text{null}(A)$. Recall that this means there are $\text{null}(A)$ essentially different linear relations among the columns of A.

$\Updownarrow$

Then the number of essentially different linear relations among the rows of A is equal to

$$m - n + \text{null}(A).$$

Let's check it. Our matrix A has $m=3$, $n=3$, & null (A), so we expect

$$3 - 3 + 1 = 1$$

relations among the rows, and this is exactly what we get. !!

In the special case of a square matrix (i.e. $m=n$) then we have

$$m - n + \text{null}(A) = \text{null}(A),$$

so the number of row relations always equals the number of column relations.

Strange But True!

A proof of the Fundamental Theorem of Linear Algebra (or even a thorough discussion of it) would lead us into a discussion of all those abstract concepts I mentioned earlier.

We won't do this. Instead we'll just accept the Fundamental Theorem as a mysterious fact of nature and we'll apply it when we need it. Our main application will be the following seemingly easy fact.

☆ Fact of Nature.

Let A & B be $n \times n$ matrices and let I be the $n \times n$ identity matrix defined by

$$I = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}.$$

<u>If</u> $AB = I$ <u>then</u> we must have $BA = I$.

[It seems easy, but it's not!]

3/16/16

HW5 : TBA.

Last time I told you about the "Fundamental Theorem of Linear Algebra". If A is a matrix, recall that we defined its nullity

$$\text{null}(A)$$

as the "dimension" of the solution of the homogeneous linear system

$$A\vec{x} = \vec{0}.$$

Equivalently, null(A) is the number of free variables in the solution, which correspond to the non-pivot columns in the RREF.

We have also seen that each of these non-pivot columns in the RREF tells us a "non-trivial linear relation" among the columns of A.

$\downarrow$

Thus we can say that

$$\text{null}(A) = \#\text{ essentially different linear relations among the columns of } A.$$

Now the Fundamental Theorem says the following.

☆ F.T.L.A.

Let A be an $m \times n$ matrix. Then the number of essentially different linear relations among the rows of A is equal to

$$m - n + \text{null}(A).$$

In particular, if A is square (i.e. $m = n$) then the number of relations among the columns of A equals the number of relations among the rows of A.

This is quite surprising! It is not obvious why relations among the rows & among the columns should have anything to do with each other.

Example from HW3:

Consider the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 & -1 & -4 \\ 1 & 2 & -1 & 4 & -1 & -4 \\ 1 & 2 & -1 & 4 & 0 & -1 \end{pmatrix},$$

which has RREF

$$\begin{pmatrix} 1 & 0 & -1 & 2 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{pmatrix}$$

There are 3 non-pivot columns, each of which corresponds to a linear relation among the columns of A. For example, the 6th column tells me that

$$1(\text{1st col.}) + (-1)(\text{2nd col.}) + 3(\text{5th col.}) = (\text{6th col.})$$

and indeed we have

$$1\begin{pmatrix}0\\1\\1\end{pmatrix}+(-1)\begin{pmatrix}1\\2\\2\end{pmatrix}+3\begin{pmatrix}-1\\-1\\0\end{pmatrix}=\begin{pmatrix}-4\\-4\\-1\end{pmatrix}.$$

Each non-pivot column also corresponds to a free variable in the solution. Thus the solution of $A\vec{x}=\vec{0}$ is a 3-plane living in $\mathbb{R}^6$ and we have

$$\text{null}(A)=3.$$

Q: How many relations are there among the <u>rows</u> of A?

A: The F.T.L.A. says there are

$$m-n+\text{null}(A)=3-6+3=0$$

relations among the rows.

In other words, the only numbers a,b,c that solve the equation

$\downarrow$

$$a\,(0\ 1\ 0\ 1\ {-1}\ {-4})$$
$$+\,b\,(1\ 2\ {-1}\ 4\ {-1}\ {-4})$$
$$+\,c\,(1\ 2\ {-1}\ 4\ 0\ {-1})$$
$$=(0\ 0\ 0\ 0\ 0\ 0)$$

is the "trivial" solution $a=b=c=0$.

[In other other words, the homogeneous matrix equation

$$(a\ b\ c)A=(0\ 0\ 0\ 0\ 0\ 0)$$

has a zero-dimensional solution. This gives you a hint that row relations come from multiplying row vectors on the left of A, while column relations come from multiplying column vectors on the right of A.].

We won't prove the F.T.L.A; we'll just accept it as a fact of nature and use it when needed.

Our main application will be to the invertibility of matrices.

Remember our goal for matrix algebra. We want to be able to do things like this:

$$A\vec{x} = \vec{b} \implies \vec{x} = A^{-1}\vec{b}.$$

If the matrix "A^{-1}" exists, what properties would it have?

- Suppose A is $m\times n$ so that $\vec{x}$ is $n\times 1$ & $\vec{b}$ is $m\times 1$. In this case the matrix A^{-1} must have shape $n\times m$:

$$\underset{n\times 1}{\vec{x}} = \underbrace{\underset{n\times \cancel{m}}{A^{-1}} \ \underset{\cancel{m}\times 1}{\vec{b}}} \qquad \checkmark$$

- Multiplying A on the left of the equation $\vec{x} = A^{-1}\vec{b}$ gives

$$\vec{b} = A\vec{x} = A(A^{-1}\vec{b}) = (AA^{-1})\vec{b},$$

and multiplying A^{-1} on the left of the equation $A\vec{x} = \vec{b}$ gives

$$\vec{x} = A^{-1}\vec{b} = A^{-1}(A\vec{x}) = (A^{-1}A)\vec{x}.$$

Thus we have

① $(AA^{-1})\vec{b} = \vec{b}$,

② $(A^{-1}A)\vec{x} = \vec{x}$.

Since these equations should hold for <u>all</u> $m \times 1$ vectors $\vec{b}$ & <u>all</u> $n \times 1$ vectors $\vec{x}$, we conclude that

$$AA^{-1} = I_m = \underbrace{\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}}_{m} \Big\} m$$

$$A^{-1}A = I_n = \underbrace{\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}}_{n} \Big\} n$$

If such a matrix A^{-1} exists, we will call it the <u>inverse matrix</u> of A.

But two questions remain:

- <u>Does</u> such a matrix A^{-1} exist?
- If A^{-1} exists, how can we <u>compute</u> it?

3/18/16

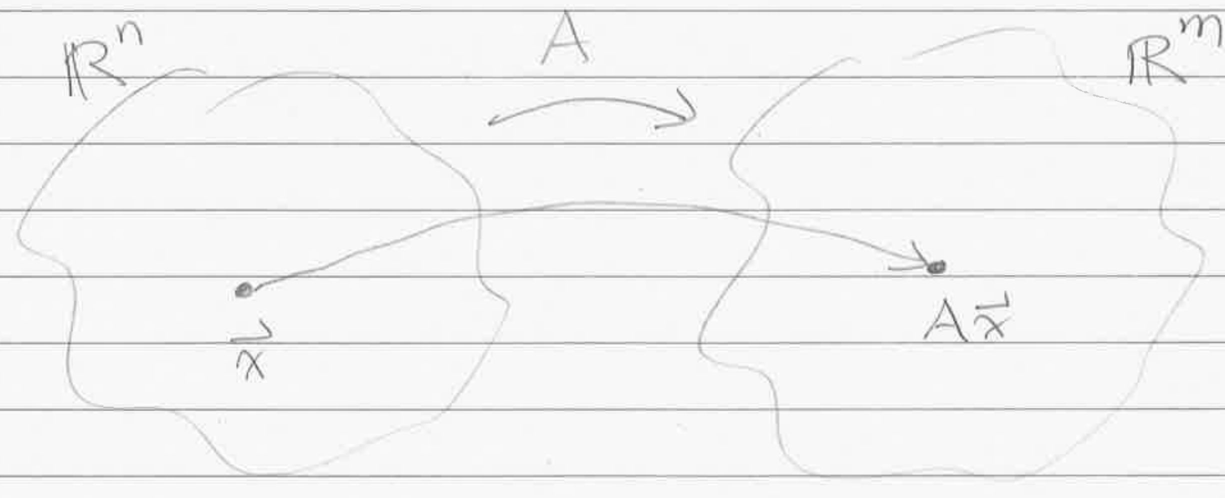
HW5 due next Fri Mar 25

Exam schedule:

- Exam 2 on Fri Apr 22 in class
- Final Exam Wed Apr 27 2-4:30pm

Today: The Inverse of a Matrix.

Let A be an $m \times n$ matrix. Recall that we can think of this as a <u>function</u> from $\mathbb{R}^n$ to $\mathbb{R}^m$:



The inverse matrix A^{-1} (<u>if</u> it exists) should be a function from $\mathbb{R}^m$ to $\mathbb{R}^n$ that "does the opposite of A".

In other words, we should have

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, bend left, "A"] \arrow[loop left, "A^{-1}A"] & \mathbb{R}^m \arrow[l, bend left, "A^{-1}"] \arrow[loop right, "AA^{-1}"] \end{tikzcd}$$

where AA^{-1} is the "do nothing function" from $\mathbb{R}^m$ to $\mathbb{R}^m$ and $A^{-1}A$ is the "do nothing function" from $\mathbb{R}^n$ to $\mathbb{R}^n$.

In matrix language we require

- A^{-1} is an $n \times m$ matrix
- $AA^{-1} = I_m$
- $A^{-1}A = I_n$,

where I_n is the identity matrix of size n,

$$I_n := \underbrace{\left.\begin{pmatrix} 1 & 0 & \cdots & & 0 \\ 0 & 1 & 0 & & \vdots \\ 0 & 0 & 1 & \ddots & 0 \\ \vdots & & & \ddots & \\ 0 & \cdots & & 0 & 1 \end{pmatrix}\right\} n}_{n}$$

The important questions are:

(1) When does A^{-1} exist?
(2) How can we compute it?

The recently discussed FTLA tells us something important about (1).

☆ Claim: If A is not square then the inverse A^{-1} does not exist.

To see this let's recall how the matrix product is computed. If A & B are matrices such that the product exists, then we have three key formulas

$$(i,j)^{th} \text{ entry of } AB = (i^{th} \text{ row of } A)(j^{th} \text{ col of } B)$$

$$i^{th} \text{ row of } AB = (i^{th} \text{ row of } A)\, B$$

$$j^{th} \text{ column of } AB = A\,(j^{th} \text{ column of } B).$$

[see HW 5 Problem 2]

The 2nd & 3rd formulas tell us that

- if A has a row relation then so does AB.
- if B has a column relation then so does AB.

Example: Let A_{i*} be the i^{th} row of A and let $(AB)_{i*}$ be the ith row of AB so the formula says

$$(AB)_{i*} = A_{i*}B .$$

Now suppose that A has some row relation, say $A_{1*} + A_{2*} = A_{3*}$. Then AB has the same row relation because

$$\begin{aligned} A_{1*} + A_{2*} &= A_{3*} \\ (A_{1*} + A_{2*})B &= A_{3*}B \\ A_{1*}B + A_{2*}B &= A_{3*}B \\ (AB)_{1*} + (AB)_{2*} &= (AB)_{3*}. \end{aligned}$$

///

Now suppose that A is $m \times n$ and B is $n \times m$ with

$$\begin{aligned} AB &= I_m \\ BA &= I_n \end{aligned}$$

We want to show that this is impossible when $m \neq n$. There are two cases.

Case 1: If $m > n$ then B is short and wide so its RREF will definitely have a non-pivot column. We conclude that

B has a column relation.

But then the product $AB = I_m$ must also have a column relation, which is impossible because the RREF of I_m is just I_m (which has no non-pivot columns).

Case 2: If $m < n$ then A is short and wide, so by the same reasoning

A has a column relation.

But then $BA = I_n$ has a column relation which is again impossible since $\text{RREF}(I_n) = I_n$ has no non-pivot columns.

This completes the proof of the claim.

Now let's discuss ②. If A is a square matrix then it may have an inverse. Let's try to compute it.

Example: Consider the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

If the inverse $A^{-1} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ exists then it must satisfy

$$(*) \quad \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Using the trick (j^{th} col AB) = A(j^{th} col B) we can break $(*)$ into two simultaneous linear systems:

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \& \quad \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and then we can (try to) solve both of the systems separately.

First System:

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \longrightarrow \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 2 & 0 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & -1 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \end{array}\right)$$

$$\longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \longrightarrow \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

Second System:

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \longrightarrow \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 2 & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 1 \end{array}\right)$$

$$\longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \longrightarrow \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

We conclude that A <u>is</u> invertible with inverse

$$A^{-1} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}.$$

[Well, there's an issue here. Certainly we know that

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

because that's the problem we were trying to solve. But it's not obvious why we should <u>also</u> have

$$\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

You should perform the multiplication to check that this is true. In general, if A & B are square matrices such that $AB = I$, then it follows from the FTLA that we must <u>also</u> have $BA = I$, but this fact is more <u>subtle</u> than most people realize!]

Remark: Hey, we used the same elimination steps for both of those linear systems. Wouldn't it be more efficient to solve them at the same time?

Sure let's just put them "next to each other" and see what happens:

$$\left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 1 \end{array}\right).$$

This cute trick can be summarized as

$$\left(A \mid I \right) \xrightarrow{\text{RREF}} \left(I \mid A^{-1} \right)$$

It might look strange, but it works (well, as long as A^{-1} exists)

3/23/16

HW5 due this Friday.

NOTICE: I'm changing the date of Exam 2 to Friday Apr 15 in class.

That way I can use the final week of classes to review for the Final Exam (which is Wed Apr 27, 2:00 - 4:30 pm).

To summarize our discussion of inverses:

Let A be an $m \times n$ matrix. We say that B is the <u>inverse</u> matrix of A if

$$AB = I_m \quad \& \quad BA = I_n.$$

Why do I say "the" inverse? Well, suppose we have another matrix C satisfying

$$AC = I_m \quad \& \quad CA = I_n.$$

Then it follows that

$$B = BI_m = B(AC) = (BA)C = I_nC = C.$$

We conclude that *if* the inverse of A exists, *then* it is unique. Since it's unique we can give it a special name:

we call it A^{-1}.

But *does* the inverse of A exist?

If A is not *square* we saw that A^{-1} *does not exist*.

So let A be square, say $m \times m$. If A^{-1} exists it will also be $m \times m$ and we can *try* to compute it with the following algorithm

$$\left(A \mid I_m \right) \xrightarrow{RREF} \left(I_m \mid A^{-1} \right).$$

The algorithm will *succeed* if and only if

$$RREF(A) = I_m.$$

In other words, the algorithm will *fail* if and only if

$$\mathrm{RREF}(A) \neq I_m.$$

or, in other words, if and only if

$$\mathrm{null}(A) \neq 0.$$

Many textbooks summarize this with a theorem of the following sort.

☆ Invertible Matrix Theorem:

Let A be a square matrix. Then the following conditions are equivalent.

- A is invertible
- $\mathrm{RREF}(A) = I$
- $\mathrm{null}(A) \neq 0$.
- A has no nontrivial column relation
- A has no nontrivial row relation.
- $\det(A) \neq 0$ [we'll discuss this later...]

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The list can be expanded depending on how much abstract nonsense you know. [The version on Wolfram MathWorld has 23 equivalent conditions!]

The important points are these:

- We know exactly when a matrix is invertible.
- We know how to compute the inverse when it exists.

Now let me summarize the basic properties of matrix algebra for future reference.

Let A, B & C be matrices and let α & β be numbers. Then the following properties hold (as long as the matrices are <u>defined</u>):

- $(\alpha+\beta)A = \alpha A + \beta A$.
- $\alpha(\beta A) = (\alpha\beta)A$.
- $\alpha(A+B) = \alpha A + \alpha B$
- $\alpha(AB) = (\alpha A)B = A(\alpha B)$
- $(A+B)C = AC + BC$
- $A(B+C) = AB + AC$
- $A+(B+C) = (A+B)+C$
- $A(BC) = (AB)C$

[The last property ("associativity" of matrix multiplication) is surprisingly useful!]

These properties generalize the properties of vector algebra & the dot product, which in turn generalize the familiar properties of addition & multiplication of numbers.

Luckily all of the properties are very intuitive. The only difference from "classical arithmetic" is that in general we have

$$AB \neq BA \;!$$

even when the matrices AB & BA are both defined and have the same shape.

Finally, let's look at the algebraic properties of inversion & transposition. Let A & B be matrices. When the following matrices exist we have

→ • $(A^{-1})^{-1} = A$
• $(A^T)^T = A$
→ • $(AB)^{-1} = B^{-1}A^{-1}$
• $(AB)^T = B^T A^T$
→ • $(A^T)^{-1} = (A^{-1})^T$
• $(A+B)^T = A^T + B^T$.

[WARNING: In general we have

$$(A+B)^{-1} \neq A^{-1} + B^{-1}$$

Indeed, if this were true then it would be true for 1×1 matrices. In other words, for all numbers a & b such that $a \neq 0$, $b \neq 0$ & $a+b \neq 0$ we would have

$$\frac{1}{a+b} = \frac{1}{a} + \frac{1}{b},$$

and you know this is not true.]

Let's examine the 1st, 3rd & 5th properties.

1st: Suppose A^{-1} exists. Then by definition we have

$$AA^{-1} = I \quad \& \quad A^{-1}A = I.$$

But these two equations <u>also</u> tell us that A is the inverse of A^{-1}:

$$A = (A^{-1})^{-1}.$$

///

3rd : Suppose A^{-1}, B^{-1} & AB exist. Then by definition we have

$$AA^{-1} = I \quad \& \quad A^{-1}A = I$$
$$BB^{-1} = I \quad \& \quad B^{-1}B = I$$

Then using the "associativity" property of matrix multiplication gives

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$$
$$= AIA^{-1} = AA^{-1} = I,$$

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B$$
$$= B^{-1}IB = B^{-1}B = I,$$

so we conclude that $B^{-1}A^{-1}$ is the inverse of AB, as desired.

5th : Suppose that A^{-1} exists, so by definition we have

$$AA^{-1} = I \quad \& \quad A^{-1}A = I.$$

Then applying the transpose to each equation gives

$$AA^{-1} = I \qquad \& \qquad A^{-1}A = I$$

$$(AA^{-1})^T = I^T \qquad (A^{-1}A)^T = I^T$$

$$(A^{-1})^T A^T = I \qquad A^T(A^{-1})^T = I \quad ,$$

which tells us that $(A^{-1})^T$ is the inverse of A^T. In other words,

$$(A^T)^{-1} = (A^T)^{-1} .$$

These "purely algebraic" properties of matrices will be useful on HW 5 Problem 4.

3/25/16

HW5 due NOW.

Today: HW5 Discussion.

Problem 1': If $A\vec{x} = \vec{b}$ & $A\vec{y} = \vec{b}$ then for all numbers t we have

$$
\begin{aligned}
&A\left(t\vec{x} + (1-t)\vec{y}\right) \\
&= A(t\vec{x}) + A((1-t)\vec{y}) \\
&= tA\vec{x} + (1-t)A\vec{y} \\
&= t\vec{b} + (1-t)\vec{b} \\
&= (\cancel{t} + 1 - \cancel{t})\vec{b} \\
&= 1\vec{b} \\
&= \vec{b}. \quad ///
\end{aligned}
$$

So what? This tells us that if a system of linear equations has two different solutions (say, $\vec{x}$ & $\vec{y}$)

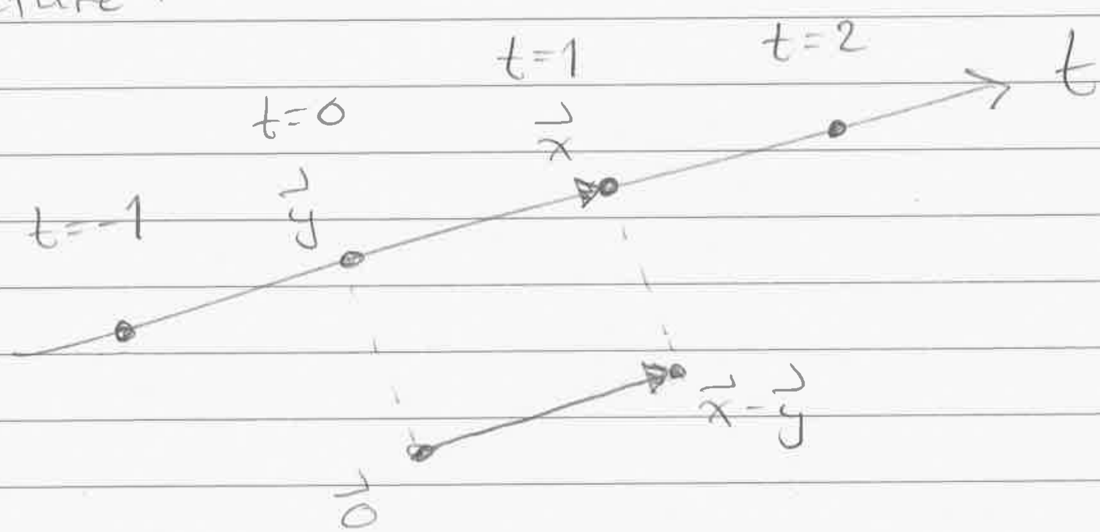
then it must have infinitely many solutions (the whole line $t\vec{x} + (1-t)\vec{y}$).

Note that

$$t\vec{x} + (1-t)\vec{y} = \vec{y} + t(\vec{x} - \vec{y}),$$

and we can think of this as the line containing the point $\vec{y}$ and parallel to the vector $\vec{x} - \vec{y}$.

Picture:



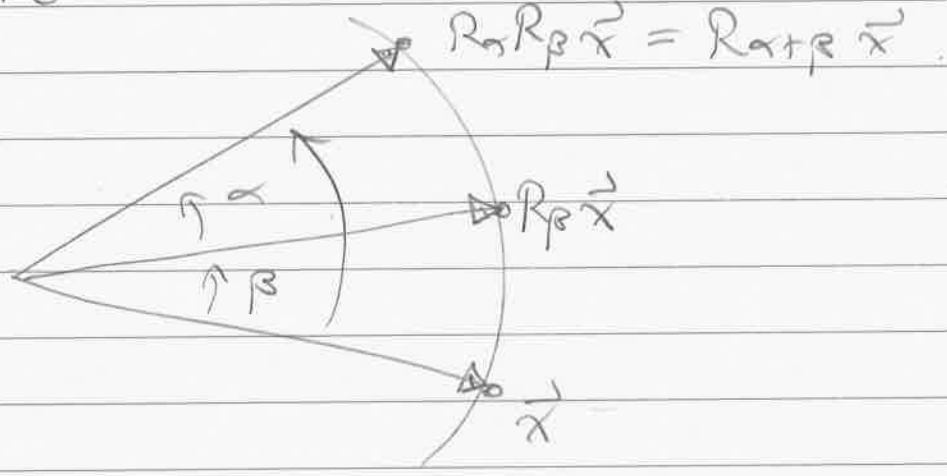
In another manner of speaking, I say that the collection of solutions of a linear system forms a "flat shape".

Problem 3': For all numbers θ, the matrix

$$R_\theta := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

defines a function $\mathbb{R}^2 \to \mathbb{R}^2$ and I claim that this is the function that rotates each vector counterclockwise by angle θ.

Now consider two angles α & β. For each vector $\vec{x}$ in $\mathbb{R}^2$ we have the following picture



Then since we have $R_\alpha R_\beta \vec{x} = R_{\alpha+\beta}\vec{x}$ for all vectors $\vec{x}$, it follows that

$R_\alpha R_\beta = R_{\alpha+\beta}$ as matrices.

So what? Expanding both sides of this equation gives

$$R_{\alpha+\beta} = R_\alpha R_\beta$$

$$\begin{pmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}$$

$$= \begin{pmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & \text{something} \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & \text{something} \end{pmatrix}.$$

Since these matrices are equal their entries must be equal, so we conclude that

$$\begin{cases} \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \end{cases}$$

You've seen these trig identities before but maybe you never knew *why* they are true. Now you know. The equation

$$R_\alpha R_\beta = R_{\alpha+\beta}$$

is the only fact about trigonometry that you need to remember.

Everything else follows from it.

Now here's a TRICK for computing the inverse of a 2×2 matrix:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

[You should check that this formula is correct.]

The number $ad-bc$ in the denominator is interesting so we will give it a name. We'll call it the <u>determinant</u> of the matrix,

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} := ad - bc .$$

Then we can see from the formula that

$$A \text{ is invertible} \iff \det(A) \neq 0 ,$$

at least when A is a 2×2 matrix.

Let's test the TRICK on the rotation matrix:

$$(R_\theta)^{-1} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}^{-1}$$

$$= \frac{1}{\cancel{(\cos\theta)^2 + (\sin\theta)^2}} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Does that make sense? Yes, because geometrically we have

$$\begin{aligned} (R_\theta)^{-1} &= (\text{rotate by } \theta \text{ counterclockwise})^{-1} \\ &= \text{rotate by } \theta \text{ clockwise} \\ &= \text{rotate by } -\theta \text{ counterclockwise} \\ &= R_{-\theta} \\ &= \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \quad \checkmark \end{aligned}$$

Finally, I'll fulfill a promise made several days ago by answering the following question.

Q: Let A & B be square matrices such that

$$AB = I.$$

It follows from this that we also have

$$BA = I,$$

but WHY?

A: This is quite subtle and most linear algebra books don't do a good job explaining it. I'll show you how the argument goes and I'll hide the hard part inside the acronym FTLA.

So assume that A & B are square with

$$AB = I.$$

Now recall that row relations in A are the same as row relations in AB.

Since $AB = I$ has no row relations we conclude that A has no row relations.

In other words, A^T has no column relations. Then the FTLA (this is the hard part) implies that A^T has no row relations.

In other words, the following row reduction will succeed:

$$\left(A^T \mid I \right) \xrightarrow{RREF} \left(I \mid C \right).$$

Now we have obtained a matrix C such that $A^T C = I$. Apply the transpose to both sides to get

$$C^T A = I.$$

Finally, we have

$$C^T = C^T I = C^T (AB) = (C^T A) B = IB = B$$

and it follows that $BA = C^T A = I$, as desired.

QED.

Remark: Yes, that is really the easiest argument that I know (and I even skipped the hard part — the FTLA).

In summary, if A is a square matrix that has a right inverse B,

$$AB = I,$$

then A must also have a left inverse C,

$$CA = I,$$

and then we must have $B = C$. We conclude that A is actually invertible with

$$A^{-1} = B.$$

You can feel free to use this fact any time, but please have the proper reverence for it.