Math 210 F
Spring 2016

Exam 1B
Friday March 3

This is a closed book test. No electronic devices are allowed. If two students submit exams in which any solution has been copied, **both students will receive a score of zero.** There are 5 pages and 5 problems, each worth 6 points.

Problem 1. Let $\vec{x}$ and $\vec{y}$ be two vectors (in some-dimensional space) such that

$$\|\vec{x}\| = 1, \qquad \|\vec{y}\| = 2, \qquad \text{and} \qquad \vec{x} \bullet \vec{y} = \sqrt{3}.$$

(a) Find the **cosine** of the angle between $\vec{x}$ and $\vec{y}$ (and the angle itself, if you know it).

$$\|\vec{x}\| \, \|\vec{y}\| \cos\theta = \vec{x} \circ \vec{y}$$

$$\cos\theta = (\vec{x} \circ \vec{y}) / (\|\vec{x}\| \, \|\vec{y}\|)$$

$$= \sqrt{3}/2 \implies \theta = \pm \frac{\pi}{6}$$

(b) Tell me the values of the dot products $\vec{x} \bullet \vec{x}$ and $\vec{y} \bullet \vec{y}$.

$$\vec{x} \circ \vec{x} = \|\vec{x}\|^2 = 1^2 = 1$$

$$\vec{y} \circ \vec{y} = \|\vec{y}\|^2 = 2^2 = 4$$

(c) Expand the expression $(\vec{y} - \vec{x}) \bullet (\vec{y} - \vec{x})$ and use the result to find the **distance between the points two points** $\vec{x}$ and $\vec{y}$.

$$\|\vec{y} - \vec{x}\|^2 = (\vec{y} - \vec{x}) \circ (\vec{y} - \vec{x})$$

$$= \vec{y} \circ \vec{y} - 2(\vec{x} \circ \vec{y}) + \vec{x} \circ \vec{x}$$

$$= 4 - 2\sqrt{3} + 1$$

$$= 5 - 2\sqrt{3}$$

$$\implies \|\vec{y} - \vec{x}\| = \sqrt{5 - 2\sqrt{3}}.$$

Problem 2. Consider the following system of 3 linear equations in 3 unknowns:

$$\begin{cases} x + y + 2z = -1 \\ x + 2y + 3z = 0 \\ 0 + y + z = 1 \end{cases}$$

(a) Put the system in reduced row echelon form (RREF).

$$\begin{array}{cc} \left(\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 1 \end{array}\right) & \begin{array}{c} A \\ B \\ C \end{array} \end{array} \rightsquigarrow \begin{array}{cc} \left(\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right) & \begin{array}{c} A \\ B-A \\ C \end{array} \end{array}$$

$$\rightsquigarrow \begin{array}{cc} \left(\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) & \begin{array}{c} A \\ B \\ C-B \end{array} \end{array} \rightsquigarrow \begin{array}{cc} \left(\begin{array}{ccc|c} 1 & 0 & 1 & -2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) & \begin{array}{c} A-B \\ B \\ C \end{array} \end{array}$$

$$\begin{cases} x + 0 + z = -2 \\ 0 + y + z = 1 \\ 0 = 0 \end{cases}$$

(b) Use your answer from part (a) to write out the **complete solution** of the system.

Let $z = t$. Then we have

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2-t \\ 1-t \\ t \end{pmatrix} = \underbrace{\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}}_{\text{This is a Line in } \mathbb{R}^3.}$$

(c) Fill in the blanks: Geometrically, this system represents three <u>planes</u> that intersect at a <u>Line</u>.

Problem 3. Now consider the modified system of 3 linear equations in 3 unknowns, where c is an arbitrary constant:

$$\begin{cases} x + y + 2z = -1 \\ x + 2y + 3z = 0 \\ 0 + y + z = c \end{cases}$$

(a) Put the system in upper-staircase form (you don't need to put it in RREF).

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & c \end{array}\right)\begin{array}{c} A \\ B \\ C \end{array} \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & c \end{array}\right)\begin{array}{c} A \\ B-A \\ C \end{array}$$

$$\rightsquigarrow \left(\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & c-1 \end{array}\right)\begin{array}{c} A \\ B \\ C \end{array} \qquad \begin{cases} x + y + 2z = -1 \\ \phantom{x+{}} y + z = 1 \\ \phantom{x+y+{}} 0 = c-1. \end{cases}$$

(b) Use part (a) to find all values of c such that the system has **no solution.**

If $c \neq 1$ then the equation $0 = c-1$ is false, so the system has no solution.

(c) If $c = 1$ then we already saw in Problem 2 that the system **has a solution.** Use your solution to express the vector $(-1, 0, 1)$ as a specific linear combination of the vectors $(1, 1, 0)$, $(1, 2, 1)$, and $(2, 3, 1)$.

For all t we have

$$(-2-t)\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + (1-t)\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + t\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

Now choose any t you want.

Problem 4. Let A be a $p \times q$ matrix (i.e. with p rows and q columns) and let B be an $m \times n$ matrix (i.e. with m rows and n columns).

(a) Fill in the blanks:

We think of A as a function from $\underline{q}$-dimensional space to $\underline{p}$-dimensional space.

We think of B as a function from $\underline{n}$-dimensional space to $\underline{m}$-dimensional space.

$$\mathbb{R}^q \xrightarrow{A} \mathbb{R}^p, \quad \mathbb{R}^n \xrightarrow{B} \mathbb{R}^m$$

(b) Finish the sentence: The product matrix AB is defined only when ...

\# columns of A = \# rows of B

$$q = m$$

(c) Fill in the blanks: If the product matrix AB is defined then we think of it as a function from $\underline{\quad n \quad}$-dimensional space to $\underline{\quad p \quad}$-dimensional space.

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "B"] \arrow[rr, bend right, "AB"'] & \overset{\mathbb{R}^q}{\mathbb{R}^m} \arrow[r, "A"] & \mathbb{R}^p \end{tikzcd}$$

(d) Finish the sentence: If the matrix AB is defined then its entry in the ith row and jth column is equal to ...

(ith row of A) $\cdot$ (jth col of B)

Problem 5. Consider the following two matrices and one vector:

$$A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & -1 \end{pmatrix}, \qquad \vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

(a) Find the vector $B\vec{x}$ by computing the dot product of $\vec{x}$ with the rows of B.

$$B\vec{x} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \begin{pmatrix} (1\ 1\ 0)\begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ (0\ 2\ -1)\begin{pmatrix} x \\ y \\ z \end{pmatrix} \end{pmatrix} = \begin{pmatrix} x+y \\ 2y-z \end{pmatrix}.$$

(b) Express the vector $A(B\vec{x})$ as a linear combination of the columns of A.

$$A(B\vec{x}) = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x+y \\ 2y-z \end{pmatrix}$$

$$= (x+y)\begin{pmatrix} 1 \\ 2 \end{pmatrix} + (2y-z)\begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} x+y \\ 2x+2y \end{pmatrix} + \begin{pmatrix} 6y-3z \\ 8y-4z \end{pmatrix} = \begin{pmatrix} x+7y-3z \\ 2x+10y-4z \end{pmatrix}$$

(c) Now find the matrix C such that for all numbers x, y, z we have $C\vec{x} = A(B\vec{x})$.

$$C\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+7y-3z \\ 2x+10y-4z \end{pmatrix}$$

$$\Longrightarrow C = \begin{pmatrix} 1 & 7 & -3 \\ 2 & 10 & -4 \end{pmatrix}.$$