HW 6: Selected Solutions

4.1.25. Suppose the vectors $\vec{a}_1, \dots, \vec{a}_n$ are unit vectors, and mutally perpendicular. That is, suppose that

$$\vec{a}_i^T \vec{a}_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

(Recall: $\vec{a}_i^T \vec{a}_i = \|\vec{a}_i\|^2$)

Now form the matrix $A = \left(\vec{a}_1 \ \vec{a}_2 \cdots \vec{a}_n \right)$.

whose jth column is $\vec{a}_j$. Then by definition, the ith row of A^T is $\vec{a}_i^T$.

We conclude that the i,j entry of A^TA is

$$(\text{ith row } A^T)\cdot(\text{jth col } A) = \vec{a}_i^T \vec{a}_j = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$$

$$\Longrightarrow \quad A^TA = \underline{I}$$

In other words, $A^T = A^{-1}$

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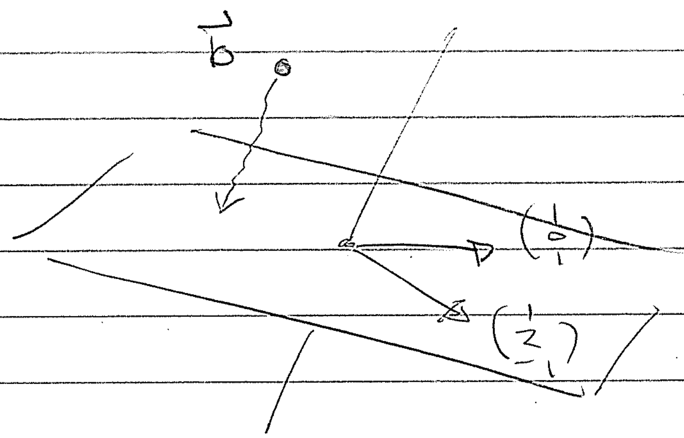
4.2.16. What linear combination of $(1,2,-1)$ and $(1,0,1)$ is closest to $(2,1,1)$?

The linear combinations $s\begin{pmatrix}1\\2\\-1\end{pmatrix} + t\begin{pmatrix}1\\0\\1\end{pmatrix}$ form a plane in $\mathbb{R}^3$.



We want to project the point $\vec{b} = (2,1,1)$ orthogonally onto the plane, and then find the s, t coordinates of the projection.

Let $A = \begin{pmatrix}1 & 1\\2 & 0\\-1 & 1\end{pmatrix}$. Then the matrix of the projection is $A(A^TA)^{-1}A^T$.

First compute $A^TA = \begin{pmatrix}1 & 2 & -1\\1 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\2 & 0\\-1 & 1\end{pmatrix} = \begin{pmatrix}6 & 0\\0 & 2\end{pmatrix}$

Then $(A^TA)^{-1} = \begin{pmatrix}1/6 & 0\\0 & 1/2\end{pmatrix} = \frac{1}{6}\begin{pmatrix}1 & 0\\0 & 3\end{pmatrix}$

Then $A(A^TA)^{-1}A = \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 1 \end{pmatrix} \frac{1}{6} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \end{pmatrix}$

$= \frac{1}{6} \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & -1 \\ 3 & 0 & 3 \end{pmatrix}$

$= \frac{1}{6} \begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & -2 \\ 2 & -2 & 4 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$

Hence the projection of $\vec{b} = (2,1,1)$ is

$\frac{1}{3} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 6 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$.

OOPS! $\vec{b}$ was already in the plane
oh well...

Find the coordinates: $s \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

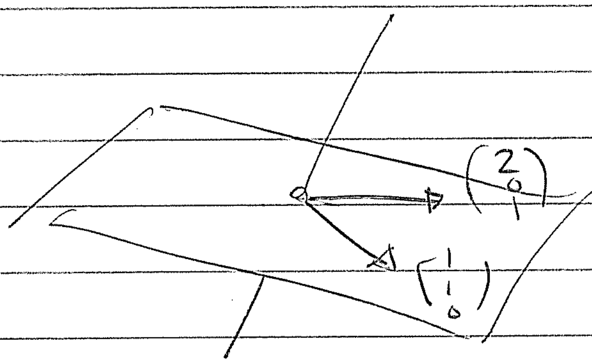
$$\begin{matrix} 1 & 1 & 2 \\ 2 & 0 & 1 \\ -1 & 1 & 1 \end{matrix} \longrightarrow \begin{matrix} 1 & 1 & 2 \\ 0 & -2 & -3 \\ 0 & 2 & 3 \end{matrix} \longrightarrow \begin{matrix} 1 & 1 & 2 \\ 0 & 1 & 3/2 \\ 0 & 0 & 0 \end{matrix} \longrightarrow \begin{matrix} 1 & 0 & 1/2 \\ 0 & 1 & 3/2 \\ 0 & 0 & 0 \end{matrix}$$

$\Longrightarrow \frac{1}{2} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \frac{3}{2} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ //

4.2.19. Project onto the plane $x-y-2z=0$.

First let $y=s$ and $z=t$. So the plane is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s+2t \\ s \\ t \end{pmatrix} = s\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$



Let $A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then the projection matrix is $A(A^TA)^{-1}A$.

First compute $A^TA = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix}$

Then $(A^TA)^{-1} = \dfrac{1}{10-4}\begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix} = \dfrac{1}{6}\begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix}$.

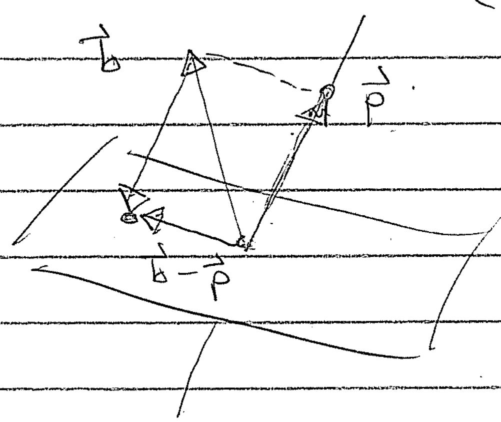
Compute $A(A^TA)^{-1}A^T$

$$= \begin{pmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{1}{6} \begin{pmatrix} \cancel{5} & \cancel{-2} \\ \cancel{-2} & \cancel{2} \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$$

$$= \frac{1}{6}\begin{pmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 5 & -2 \\ 2 & -2 & 2 \end{pmatrix}$$

$$= \frac{1}{6}\begin{pmatrix} 5 & 1 & 2 \\ 1 & 5 & -2 \\ 2 & -2 & 2 \end{pmatrix}$$

4.2.20. Another way to compute the projection. The plane $x - y - 2z = 0$ has normal vector $(1, -1, -2)$.



To project any $\vec{b}$ onto the plane, first compute the projection $\vec{p}$ of $\vec{b}$ onto the line $(1, -1, -2)$. Then the projection onto the plane is $\vec{b} - \vec{p}$.

If P projects on the plane and Q projects on the normal line, then

$$P\vec{b} = \vec{b} - \vec{p} = I\vec{b} - Q\vec{b} = (I-Q)\vec{b}$$

Hence $\boxed{P = I - Q}$.

But we know how to compute Q:

$$Q = \frac{\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}(1\ -1\ -2)}{(1\ -1\ -2)\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}} = \frac{1}{6}\begin{pmatrix} 1 & -1 & -2 \\ -1 & 1 & 2 \\ -2 & 2 & 4 \end{pmatrix}$$

Hence

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \frac{1}{6}\begin{pmatrix} 1 & -1 & -2 \\ -1 & 1 & 2 \\ -2 & 2 & 4 \end{pmatrix}$$

$$= \frac{1}{6}\begin{pmatrix} 5 & 1 & 2 \\ 1 & 5 & -2 \\ 2 & -2 & 2 \end{pmatrix}$$

We got the same answer as 4.2.19