Quiz 1 Discussion:

Given $\vec{x} \circ \vec{x} = \vec{y} \circ \vec{y} = 1$ & $\vec{x} \circ \vec{y} = 0$, find the angle between $\vec{x} + \vec{y}$ & $\vec{x} - \vec{y}$.

Say $\vec{u} = \vec{x} + \vec{y}$

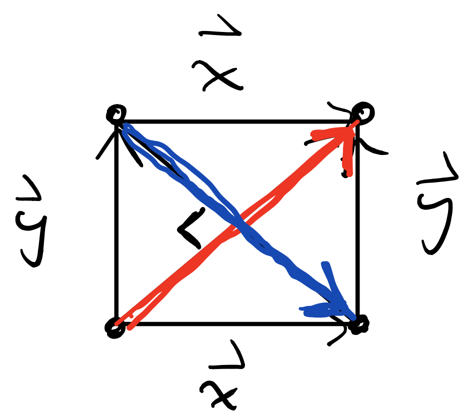
$\vec{v} = \vec{x} - \vec{y}$.

$$\begin{aligned}
\vec{u} \circ \vec{v} &= (\vec{x} + \vec{y}) \circ (\vec{x} - \vec{y}) \\
&= \vec{x} \circ \vec{x} + \vec{x} \circ \vec{y} - \vec{x} \circ \vec{y} - \vec{y} \circ \vec{y} \\
&= 1 + 0 - 0 - 1 \\
&= 0.
\end{aligned}$$

Hence $\cos\theta = \dfrac{0}{\|\vec{u}\| \|\vec{v}\|} = 0$.

So $\theta = 90°$.

Picture:



$\vec{x} + \vec{y}$ & $\vec{x} - \vec{y}$ are diagonals of a square, hence perpendicular.

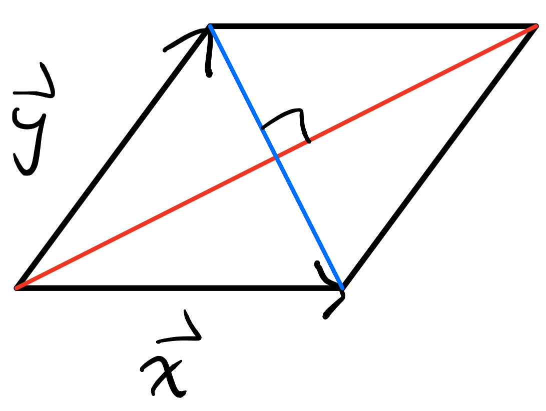
More generally, let $\vec{x}$ & $\vec{y}$ be any vectors of <u>the same length</u>, so

$$\|\vec{x}\| = \|\vec{y}\|$$
$$\|\vec{x}\|^2 = \|\vec{y}\|^2$$
$$\vec{x}\cdot\vec{x} = \vec{y}\cdot\vec{y}$$

Then we have

$$(\vec{x}+\vec{y})\cdot(\vec{x}-\vec{y})$$
$$= \cancel{\vec{x}\cdot\vec{x}} + \cancel{\vec{x}\cdot\vec{y}} - \cancel{\vec{x}\cdot\vec{y}} - \cancel{\vec{y}\cdot\vec{y}}$$
$$= 0.$$

Picture:



Theorem: The diagonals in a "rhombus" are perpendicular.

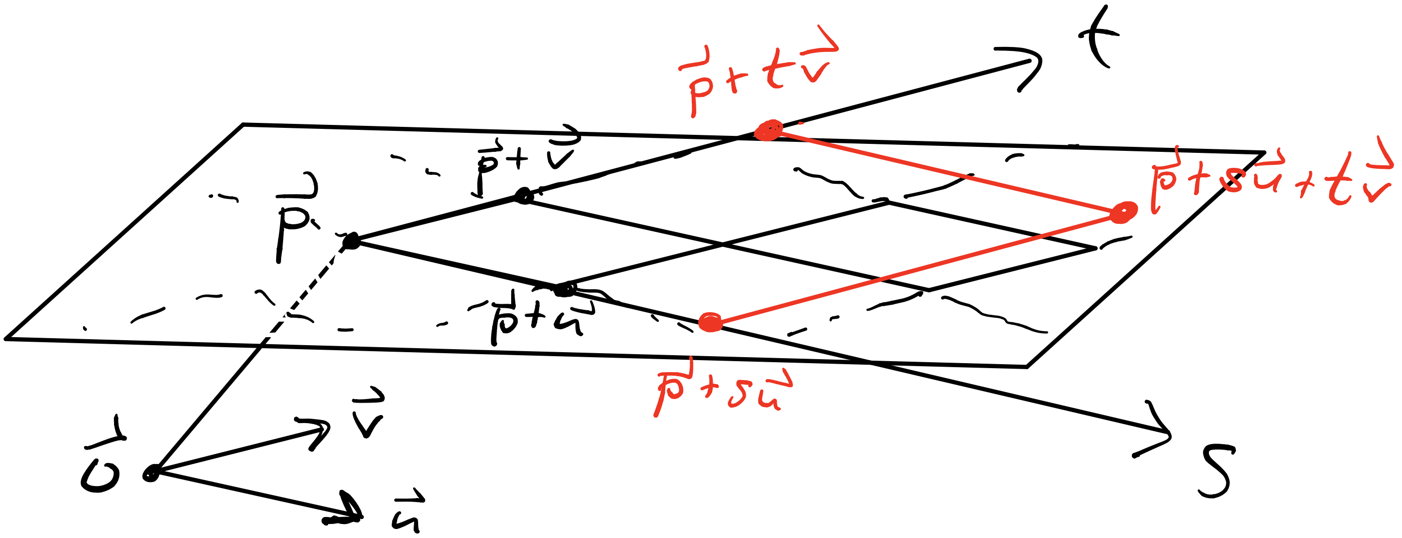
New Topic: Planes in $\mathbb{R}^3$
& Introduction to Linear Systems.

How to describe a <u>plane</u> in $\mathbb{R}^3$?

- A point $\vec{p}$
  & two direction vectors $\vec{u}$ & $\vec{v}$.

Then points $\vec{x}$ on the plane have the form $\boxed{\vec{x} = \vec{p} + \underbrace{s\vec{u} + t\vec{v}}_{\text{two parameters}}.}$

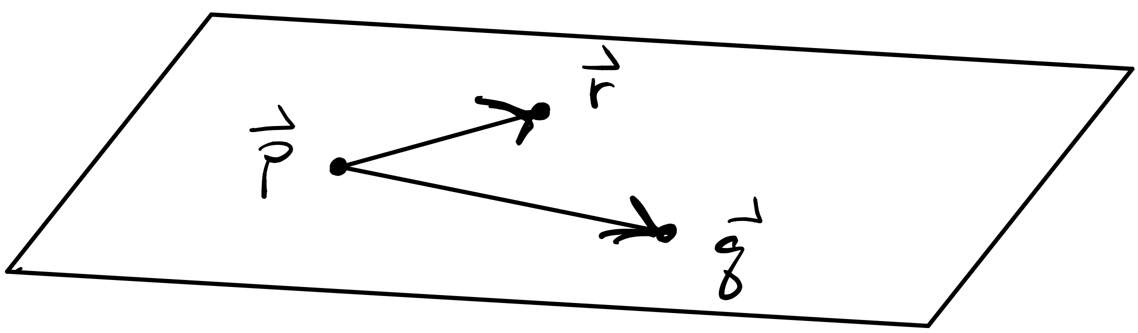
Picture:



Every point in the plane has the form "$\vec{p} + s\vec{u} + t\vec{v}$" for some unique values of s & t. We could call this the "(s,t)-plane."

We can think of $\vec{p}$, $\vec{u}$ & $\vec{v}$ as a "coordinate system" for the plane.

- Three points $\vec{p}, \vec{q}, \vec{r}$.



We can take
$$\vec{u} = \vec{q} - \vec{p}$$
$$\vec{v} = \vec{r} - \vec{p}$$
as direction vectors, so that

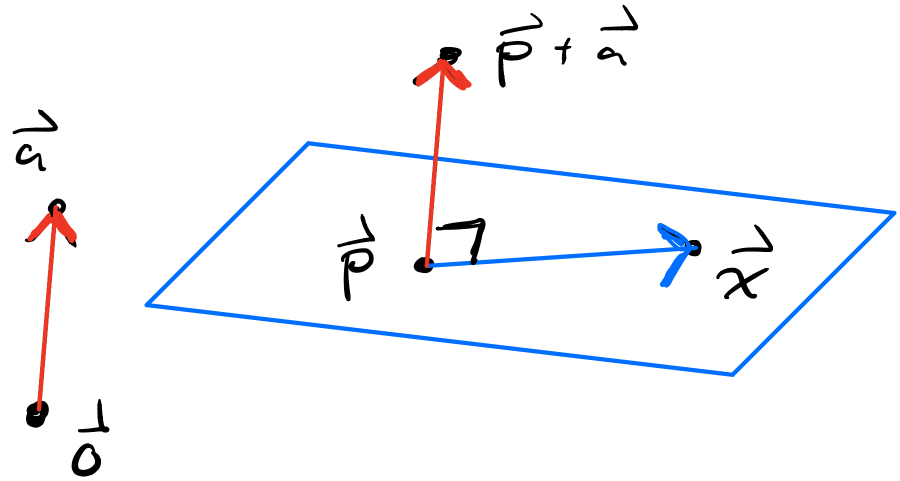
$$\text{plane} = \{\vec{p} + s\vec{u} + t\vec{v} : s,t \in \mathbb{R}\}$$
$$= \{\vec{p} + s(\vec{q} - \vec{p}) + t(\vec{r} - \vec{p}) : s,t \in \mathbb{R}\}$$

$$= \{ \underbrace{\underbrace{(1-s-t)}\vec{p} + s\vec{q} + t\vec{r}}_{\text{coeffs. add to } 1.} : s,t \in \mathbb{R} \}$$

$$= \{ x\vec{p} + y\vec{q} + z\vec{r} : x+y+z = 1 \}$$

[Once you've seen lines in $\mathbb{R}^2$ and planes in $\mathbb{R}^3$ you can easily imagine how these formulas generalize to higher dimensions ...]

- A point $\vec{p}$ & a normal vector $\vec{a}$.



For any point $\vec{x}$ on the plane we require that vector $\vec{x}-\vec{p}$ is perpendicular to vector $\vec{a}$:

$$\boxed{\vec{a} \circ (\vec{x} - \vec{p}) = 0.}$$

$$\vec{a} \circ \vec{x} - \vec{a} \cdot \vec{p} = 0$$

$$\vec{a} \circ \vec{x} = \vec{a} \circ \vec{p}$$

$$\vec{a} \circ \vec{x} = \text{constant.}$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \circ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \text{const.}$$

$$\boxed{ax + by + cz = \text{const.}}$$

Examples :

Express the plane $x + 2y + 3z = 1$ in the "parametric form"

$$\vec{x} = \vec{p} + s\vec{u} + t\vec{v}.$$

Let's take $\vec{p} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ ok ✓

Tell me 2 more points on the plane.

$\vec{q} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$ & $\vec{r} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ ✓

Let $\vec{u} = \vec{q} - \vec{p} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$

$\vec{v} = \vec{r} - \vec{p} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$

be direction vectors. Then

$$\vec{x} = \vec{p} + s\vec{u} + t\vec{v}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} + t\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{cases} x = 1 - s - 2t, \\ y = -s + t, \\ z = s. \end{cases}$$

But this answer is <u>not unique</u>.

There are ∞ many ways to parametrize a plane.

Quick Method (Shortcut):

Let $y = s$ & $z = t$. Then

$$\boxed{x + 2y + 3z = 1}$$ Definition of the plane.

$$x + 2s + 3t = 1$$

$$x = 1 - 2s - 3t.$$

$$\begin{cases} x = 1 - 2s - 3t, \\ y = s, \\ z = t. \end{cases}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 - 2s - 3t \\ s \\ t \end{pmatrix}$$

$$= \begin{pmatrix} 1 - 2s - 3t \\ 0 + 1s + 0t \\ 0 + 0s + 1t \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}.$$

So $\vec{u} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$ & $\vec{v} = \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$ is

another valid choice of direction vectors for the plane.

On Thurs we will begin with more examples....