Quiz 1 : Average 8.09/10

Quiz 2 : Average 7.9/10

See solutions online.

HW 3 due this Thurs before class.

Current Topic: Gaussian Elimination (AKA Row Reduction) and the Reduced Row Echelon Form (RREF) of a matrix.

Def: A matrix is a rectangle of numbers

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

If A has m rows & n columns, we say A has "shape $m \times n$" (say the # of rows first).

[The name "matrix" comes from J.J. Sylvester (~1860): a womb that gives birth to determinants.]

Example from last time:

$$\begin{cases} x + y = 1 \\ x + 2y = -1 \\ 2x + 3y = 0 \end{cases} \rightsquigarrow \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 2 & 3 & 0 \end{array}\right)$$

[The vertical line is just to help us rember that this matrix came from a system of linear equations.]

We performed a sequence of EROs:

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 2 & 3 & 0 \end{array}\right) \overset{\text{EROS}}{\rightsquigarrow} \left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array}\right) \rightsquigarrow \begin{cases} x + 0 = 3 \\ 0 + y = -2 \\ 0 + 0 = 0 \end{cases}$$

An equation of the form $0 = 0$ is always true (we call it "redundant") so we can just throw it away!

$$\begin{array}{l} ① \\ ② \\ ③ \end{array} \left\{\begin{array}{l} x + y = 1 \\ x + 2y = -1 \\ 2x + 3y = 0 \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{l} x + 0 = 3 \\ 0 + y = -2 \end{array}\right\}$$

It looked like we had 3 equations but we really only had 2.

To be specific, there was a nontrivial relation ("redundancy") among the original equations:

$$① + ② = ③ \qquad (*)$$

Usually that wouldn't happen for a matrix of this shape.

Example: If we perturb the matrix slightly (change entries by small amounts), then we will kill the relation (*) and we won't create any new relations. Let's change the bottom right entry ($c \neq 0$):

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 2 & 3 & c \end{array}\right) \xrightarrow{\text{EROS}} \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)$$

System is equivalent to

$$\begin{cases} x + 0 = 0, \\ 0 + y = 0, \\ 0 + 0 = 1. \end{cases}$$

The equation $0 + 0 = 1$ is NEVER true, so system has NO SOLUTION.

Summary: If EROs create a row of form $0\ 0 \cdots 0 \mid 0$ then we can throw this row away because it gives no information. [It happened because there was a nontrivial relation among the rows.]

If EROs create a row of the form $0\ 0 \cdots 0 \mid c$ where $c \neq 0$, then you can stop because the system has <u>no solution</u>.

Jargon:

"consistent" = "has a solution"

"inconsistent" = "has no solution"

Now: Gaussian Elimination Algorithm for Computers.

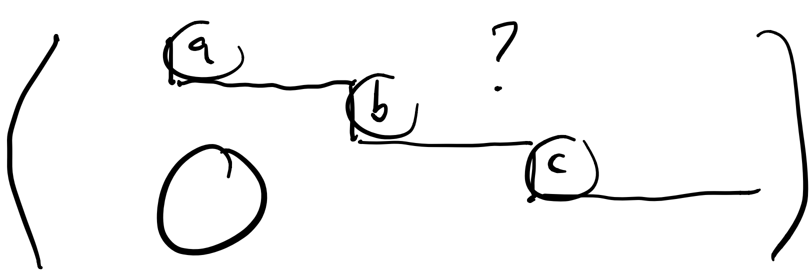
Input: matrix A (shape $m \times n$)

Algorithm has 3 steps.

Step 1:

- move to leftmost nonzero column.
- swap rows (ERO Type I) to get nonzero entry in top row of this column. Call this the "pivot" entry.
- Eliminate all entries below the pivot entry. (EROs Type III)
- Delete / ignore 1st row and repeat on the remaing rows.

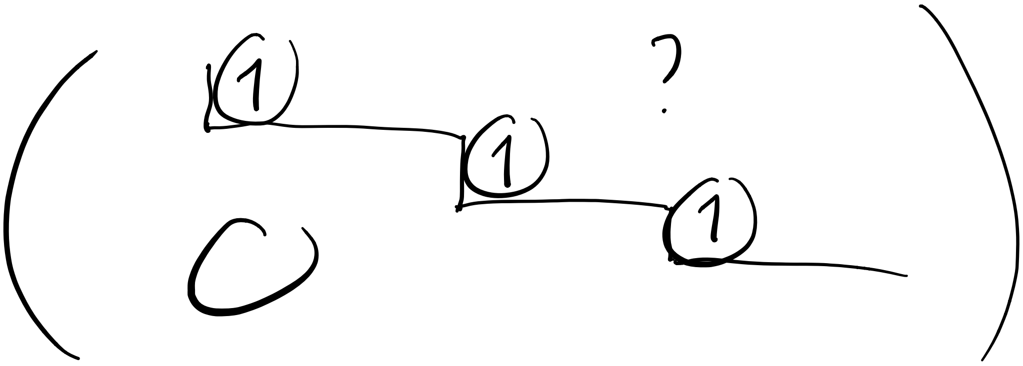
Result:



$a, b, c \neq 0$.

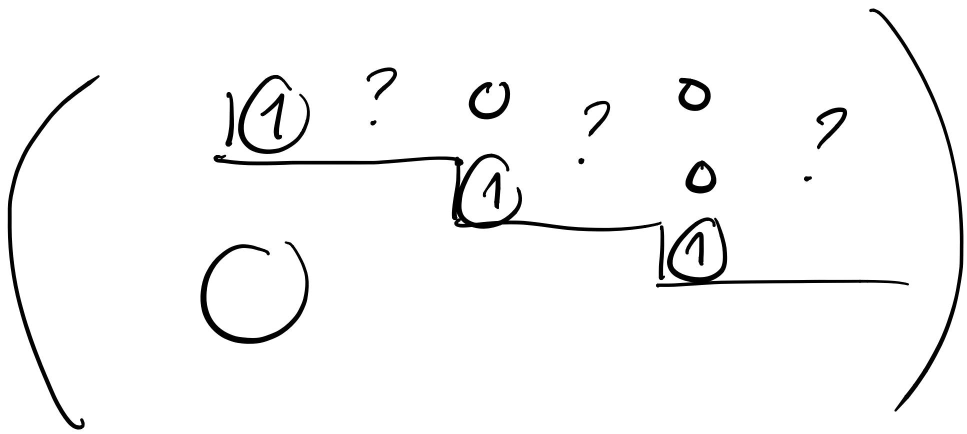
Step 2: Scale the rows (EROs of type II) to convert each pivot entry to 1.

Result:



Step 3: Apply EROs of Type III to eliminate all entries above the pivot entries.

Result:



STOP

Output: RREF(A), the Reduced Row Echelon Form of A.

Let's interpret RREF(A) when A is $m \times (n+1)$ matrix coming from a system of m linear equations in n unknowns $x_1, x_2, \dots, x_n$.

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = b_1, \\ a_{21}x_1 + \cdots + a_{2n}x_n = b_2, \\ \quad\vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m. \end{cases}$$

$$A = \underbrace{\left(\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right)}_{n+1 \text{ columns}} \left.\vphantom{\begin{array}{c} a \\ \vdots \\ a \end{array}}\right\} \begin{array}{l} m \\ \text{rows} \end{array}$$

Compute RREF(A):

$$\left(\begin{array}{ccccccc|c} 0 & 1 & a & b & 0 & c & 0 & e \\ & & & & 1 & d & 0 & f \\ & & & & & & 1 & g \\ \hline & & 0 & & & & & 0 \end{array}\right)$$

Pivot entries correspond to pivot variables. In this example:

$$x_2, x_5, x_7$$

All other variables are free:

$$x_1, x_3, x_4, x_6.$$

If a pivot entry occurs in final column then there is no solution.

Jargon:

Let $r = \#$ pivots in $RREF(A)$

Called the "rank" of matrix A.

Let d = dimension of the set of solutions, i.e., the solutions are a d-plane in $\mathbb{R}^n$.

This means there are d pivot variables. Let's rename them

$$t_1, t_2, \dots, t_d.$$

General solution is

$$\vec{x} = \vec{p} + t_1 \vec{u}_1 + \cdots + t_d \vec{u}_d$$

where $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_d$ are independent.

We observe that

$$r = \#\text{ pivot variables}$$

$$d = \#\text{ non-pivot variables}$$

$$\Longrightarrow \quad r + d = \#\text{ variables} = n$$

$$\boxed{r + d = n}$$

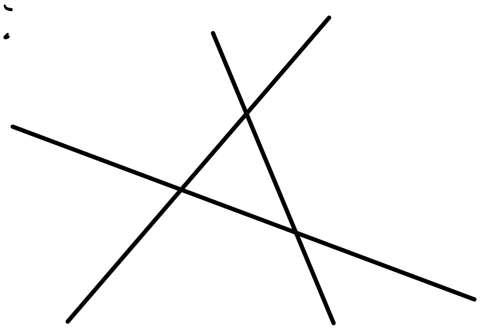
This formula is sometimes called the "Rank-Nullity Theorem."

Let's do some examples on the computer.

- Shape 3×3:

$$\left(\begin{array}{cc|c} 0 & 2 & 0 \\ 1 & 1 & -1 \\ -1 & 2 & -1 \end{array}\right) \xrightarrow{\text{RREF}} \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)$$

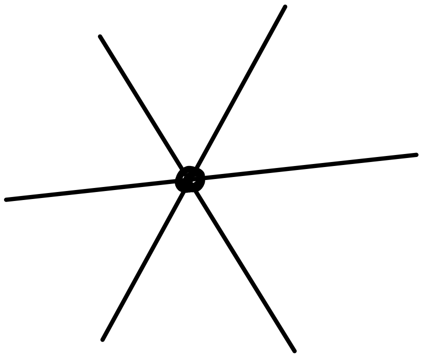
If we interpret as a system of 3 lines in $\mathbb{R}^2$, there is no solution:



- Now give me a 3×3 matrix with a row relation (say ② = ① + 2③)

$$\begin{pmatrix} 2 & 2 & 2 \\ 4 & 8 & 10 \\ 1 & 3 & 4 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 3/2 \\ 0 & 0 & 0 \end{pmatrix}$$

If we interpret this as a system of 3 lines in $\mathbb{R}^2$ then the lines meet at a point



point of intersection is $(x,y) = \left(\frac{-1}{2}, \frac{3}{2}\right)$.

This won't happen by accident; we caused it to happen by forcing a row relation.

- 4×7 matrix

$$\begin{pmatrix} 7 & 8 & 2 & 3 & 1 & 4 & 5 \\ 3 & 2 & 1 & -2 & 0 & 5 & 6 \\ 1 & 2 & 1 & 0 & 2 & 1 & 4 \\ 4 & 1 & 5 & 3 & 6 & 2 & 4 \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} 1 & 0 & 0 & 0 & -41/40 & 42/40 & -26/40 \\ 0 & 1 & 0 & 0 & 22/40 & -4/40 & 52/40 \\ 0 & 0 & 1 & 0 & 77/40 & 6/40 & 82/40 \\ 0 & 0 & 0 & 1 & -1/40 & -38/40 & \frac{-66}{40} \end{pmatrix}$$

This is exactly what we expected:

- a pivot in every row
- First columns got the pivots
- a bunch of fractions.

If it came from system of 4 eqns in 6 unknowns $x_1, \ldots, x_6$, then the solution is

$$\vec{x} = \vec{p} + x_5 \vec{u} + x_6 \vec{v}$$

2-plane living in $\mathbb{R}^6$