HW3 due next Thurs, Oct 1.

Review:

- one linear equation in n unknowns represents a hyperplane ($(n-1)$-dimensional plane) in $\mathbb{R}^n$.
- m linear equations in n unknowns represents the intersection of m hyperplanes in $\mathbb{R}^n$.
- This intersection is either
  - empty, or
  - a d-dimensional plane for some $n-m \le d \le n$
- If the equations are chosen "randomly" (Technically: If the normal vectors of the hyperplanes are "independent")

then we have

$$d = n - m$$

dim of solutions = # variables − # equations

- Example $n = 3$:

Assuming that solution is not empty,

$m = 1 \rightsquigarrow d = 2$

$m = 2 \rightsquigarrow d = 1$ or 2

[Two planes intersect in a line or in a plane.]

$m = 3 \longrightarrow d = 0$ or 1 or 2

[Three planes intersect in a point, or a line, or a plane.]

$m \geq 4 \rightsquigarrow$ probably no solution

- Example $n = 4$, $m = 2$:

$$\begin{cases} x_1 + 3x_2 + 0 + 2x_4 = 1, \\ x_1 + 3x + x_3 + 6x_4 = 7. \end{cases}$$

Solution is a 2-plane in $\mathbb{R}^4$:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ 0 \\ 6 \\ 0 \end{pmatrix}}_{\text{point}} + \underbrace{s \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ 0 \\ -1 \\ 1 \end{pmatrix}}_{\text{two independent direction vectors}}$$

point

two independent direction vectors.

Now the general method:

"Gaussian Elimination"

[History: Gauss invented this method around 1800 in order to predict the orbit of the dwarf planet Ceres.]

I'll illustrate the method with an example $(n=4, m=3)$:

$$\begin{cases} ① & x_1 + 0 + 3x_3 - x_4 = -3, \\ ② & x_1 + x_2 + x_3 + 0 = 3, \\ ③ & x_1 + 2x_2 - x_3 + 2x_4 = 8. \end{cases}$$

First get rid of unnecessary symbols:

$$\left(\begin{array}{cccc|c} 1 & 0 & 3 & -1 & -3 \\ 1 & 1 & 1 & 0 & 3 \\ 1 & 2 & -1 & 2 & 8 \end{array}\right) \begin{array}{c} 1 \\ 2 \\ 3 \end{array}$$

Now we will perform a sequence of "elementary row operations" (EROs):

Type I: Swap two rows

Type II: Scale a row

$$i \longmapsto a\, i \qquad (a \neq 0)$$

Type III: Add a scalar multiple of one row to another

$$i \longmapsto i + a\, j \qquad ///$$

Key Fact: EROs preserve the solutions of a system:

$$\{\text{system } 1\} \xrightarrow{\text{EROs}} \{\text{system } 2\}$$

Then these two systems have the same set of solutions.

Goal: Perform EROs in order to simplify as much as possible, so then the solution will be easy to see. Jargon:

$$\left\{ \begin{array}{c} \text{original} \\ \text{system} \end{array} \right\} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \left\{ \text{RREF} \right\}$$

RREF = "Reduced Row Echelon Form of the system"

Our Example:

$$\begin{array}{c} \left(\begin{array}{cccc|c} 1 & 0 & 3 & -1 & -3 \\ 1 & 1 & 1 & 0 & 3 \\ 1 & 2 & -1 & 2 & 8 \end{array} \right) \end{array} \begin{array}{c} ① \\ ② \\ ③ \end{array}$$

The top left entry is our first "pivot" entry. We perform EROs of Type III to eliminate all entries below the pivot:

$$\left(\begin{array}{cccc|c} 1 & 0 & 3 & -1 & -3 \\ 0 & 1 & -2 & 1 & 6 \\ 0 & 2 & -4 & 3 & 11 \end{array}\right) \quad \begin{array}{l} (1) \\ (2) \to (2) - 1(1) \\ (3) \to (3) - 1(1) \end{array}$$

Top left entry below the first row is the new pivot enty. Perform EROs of Type III to eliminate below the new pivot:

$$\left(\begin{array}{cccc|c} 1 & 0 & 3 & -1 & -3 \\ 0 & 1 & -2 & 1 & 6 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right) \quad \begin{array}{l} (1) \\ (2) \\ (3) \to (3) - 2(2) \end{array}$$

?

Now we try to find a pivot in the blue position, but the entry is zero so we move on to the right.

We find our next pivot in the 3rd row, 4th column:

$$\left(\begin{array}{cccc|c} 1 & 0 & 3 & -1 & -3 \\ 0 & 1 & -2 & 1 & 6 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right) \begin{array}{c} ① \\ ② \\ ③ \end{array}$$

Now the sytem is in EF ("Echelon Form"):

- Zeroes below a staircase.
- Nonzero entries in the corners of the staircase, called the "pivot entries."

Next step: To put system in REF ("Reduced Echelon Form"), scale the rows (EROs of Type II) to turn all pivot entries into 1's.

[Already Done ✓].

Final step: Perform EROs of Type III to eliminate all entries

ABOVE the pivots:

$$\left(\begin{array}{cccc|c} 1 & 0 & 3 & 0 & -4 \\ 0 & 1 & -2 & 0 & 7 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right) \begin{array}{l} ① \to ① + 1③ \\ ② \to ② - 1③ \\ ③ \end{array}$$

Now we're done! This is called the RREF ("Reduced Row Echelon Form"). Let's convert this back into a system of linear equations:

$$\begin{cases} x_1 + 0 + 3x_3 + 0 = -4, \\ 0 + x_2 - 2x_3 + 0 = 7, \\ 0 + 0 + 0 + x_4 = -1. \end{cases}$$

To read the solution:

The circled entries are the PIVOT VARIABLES: x_1, x_2, x_4.

The non-pivot variables are FREE.

I like to rename them, say $x_3 = t$.

Thus we have

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -4-3t \\ 7+2t \\ t \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} -4-3t \\ 7+2t \\ 0+1t \\ -1+0t \end{pmatrix} = \begin{pmatrix} -4 \\ 7 \\ 0 \\ -1 \end{pmatrix} + t \begin{pmatrix} -3 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$

Geometric interpretation: 3 hyperplanes in $\mathbb{R}^4$, intersecting in a <u>line</u>. [Can you vizualize this? No you can't.]

This is as expected because

$$\#\text{variables} - \#\text{equations}$$

$$= 4 - 3 = 1 \quad (\text{a line!}).$$

Look at the computer [MAPLE]
to check our work.

Let's look at an example
where the expected thing does not
happen:

$$\begin{cases} x + y = 1, & (1) \\ x + 2y = -1, & (2) \\ 2x + 3y = 0. & (3) \end{cases}$$

What do we expect?

$\#\text{unknowns} - \#\text{equations}$

$= 2 - 3 = -1 \ (?)$

We expect NO SOLUTION.

But let's see what happens...

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 2 & 3 & 0 \end{array}\right) \quad \begin{array}{l} (1) \\ (2) \\ (3) \end{array}$$

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \end{array}\right) \quad \begin{array}{l} (1) \\ (2) \rightsquigarrow (2) - 1(1) \\ (3) \rightsquigarrow (3) - 2(1) \end{array}$$

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array}\right) \quad \begin{array}{l} (1) \\ (2) \\ (3) \rightsquigarrow (3) - 1(2) \end{array}$$

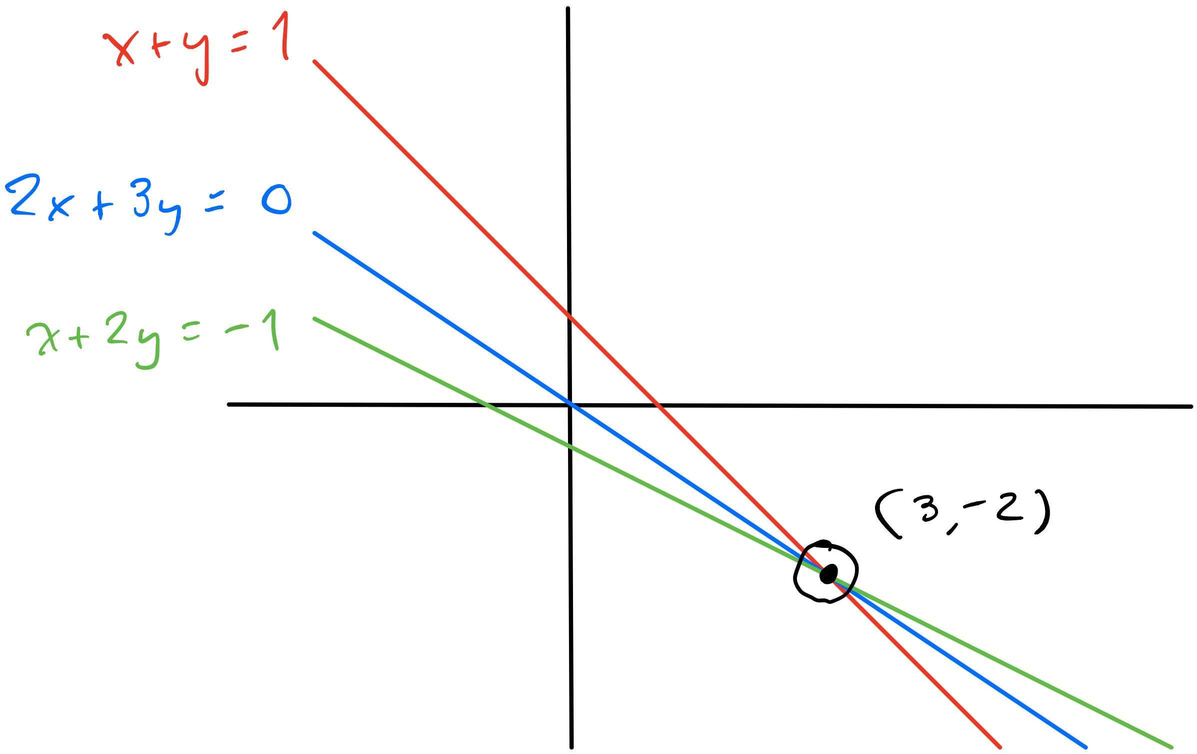
$$\left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array}\right) \quad \begin{array}{l} (1) \rightsquigarrow (1) - 1(2) \\ (2) \\ (3) \end{array}$$

Done ✓ Convert back to a system:

$$\begin{cases} x + 0 = 3, \\ 0 + y = -2, \\ \boxed{0 + 0 = 0}. \end{cases}$$

Third equation is true for <u>any</u> values of x & y, so we can ignore it.

The solution is a point $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$.



This happened because our original system had an unexpected linear relation among the rows:

$$(1) + (2) = (3).$$

If we perturb the system slightly then will kill this relation, and we won't create any new relations.

For example, consider the system

$$\begin{array}{l} ① \\ ② \\ ③ \end{array} \begin{cases} x + y = 1, \\ x + 2y = -1, \\ 2x + 3y = c. \end{cases}$$

After performing the same EROs as before, this becomes

$$\begin{cases} x + 0 = 3, \\ 0 + y = -2, \\ \boxed{0 + 0 = c}. \end{cases}$$

If $c \neq 0$ then the 3rd equation

$$0x + 0y = c$$

is always false, hence this system has no solution.