Previous Topic: Systems of m linear equations in n unknowns when $n = 2$ (lines in $\mathbb{R}^2$) or $n = 3$ (planes in $\mathbb{R}^3$).

New Topic: General n!

e.g. 55 linear equations in 78 unknowns. Geometrically, this is an intersection of 55 hyperplanes in 78-dimensional space. Believe it or not, humans have a complete and satisfactory understanding of this problem.

Before giving the algorithm, we need a bit of jargon to describe the solutions.

JARGON:

- Given vectors $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_d \in \mathbb{R}^n$, a linear combination is any expression

"$t_1\vec{u}_1 + t_2\vec{u}_2 + \cdots t_d\vec{u}_d$"

for scalars $t_1, t_2, \dots, t_d \in \mathbb{R}$.

- We say vectors $\vec{u}_1, \dots, \vec{u}_d \in \mathbb{R}^n$ are dependent if we have a nontrivial linear relation:

$$t_1\vec{u}_1 + t_2\vec{u}_2 + \cdots + t_d\vec{u}_d = \vec{0},$$

where the t_i are not all zero.

- Independent $\equiv$ not dependent
$\equiv$ no nontrivial relation

Idea: We say $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_d$ are independent if they point in "d different directions"

Examples:

- $\vec{u}, \vec{v} \in \mathbb{R}^2$ are independent

  $\iff$ not parallel.

  $\iff$ $\det(\vec{u}\ \vec{v}) \neq 0$.

- $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$ are independent

  $\iff$ not coplanar

  (i.e. do not lie in same plane)

  $\iff$ $\det(\vec{u}\ \vec{v}\ \vec{w}) \neq 0$. ///

- If vectors $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_d \in \mathbb{R}^n$ are independent, then for any point $\vec{p} \in \mathbb{R}^n$, the set

$$\vec{x} = \vec{p} + t_1\vec{u}_1 + t_2\vec{u}_2 + \dots + t_d\vec{u}_d$$

is called a "d-dimensional plane," (or just a "d-plane") living in n-dimensional space $\mathbb{R}^n$.

Common Names:

0-plane $\equiv$ point $(\vec{x} = \vec{p})$

1-plane $\equiv$ line $(\vec{x} = \vec{p} + t\vec{u})$

2-plane $\equiv$ plane $(\vec{x} = \vec{p} + s\vec{u} + t\vec{v})$

$\vdots$

(n-1)-plane $\equiv$ hyperplane

n-plane $\equiv$ whole space $\mathbb{R}^n$.

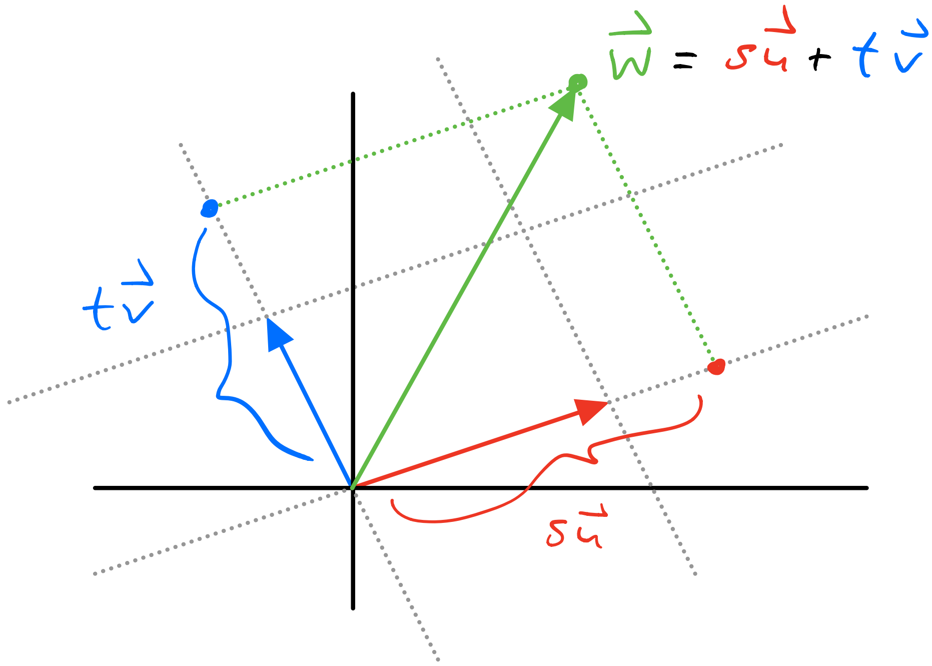
Remark: If $d > n$, then it is <u>not possible</u> to find d independent vectors in $\mathbb{R}^n$.

Example: Any set of 3 vectors $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^2$ in the plane must be <u>dependent</u>, i.e., have a relation

$$r\vec{u} + s\vec{v} + t\vec{w} = \vec{0}.$$

Say $\vec{u} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ & $\vec{v} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$.

But then, any other vector $\vec{w} \in \mathbb{R}^2$ must have the form $\vec{w} = s\vec{u} + t\vec{v}$ for some scalars $s, t \in \mathbb{R}$:



Hence the vectors $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^2$ satisfy a nontrivial relation:

$$-s\vec{u} - t\vec{v} + 1\vec{w} = \vec{0}.$$

This is what we mean when we say that the plane $\mathbb{R}^2$ is

"2-dimensional."

This jargon allows me to state

The Dimension Principle:

The solution of m linear equations in n unknowns is either:

- empty, or
- a d-plane for some

$$n-m \leq d \leq n.$$

If $m \leq n$ and equations are chosen randomly, then we will have

$$d = n - m$$

$$\boxed{\text{dim of solution} = \#\text{variables} - \#\text{equations}}$$

Idea: Every new equation (probably) cuts down the dimension by 1.

If $m > n$ ($\#$ equations $>$ $\#$ unknowns) then there is (probably)

NO SOLUTION.

We've already seen how this works when $n = 2$ or 3, so let's consider an example in $n = 4$ variables.

Example: Solve for $\vec{x} = (x_1, x_2, x_3, x_4)$:

$$\begin{cases} x_1 + 3x_2 + 0 + 2x_4 = 1, & (1) \\ x_1 + 3x_2 + x_3 + 6x_4 = 7. & (2) \end{cases}$$

What do we expect?

$n = 4$ variables,

$m = 2$ equations,

so the solution is probably a <u>2-plane</u>.

$$\text{dim of solution} = \#\text{ variables} - \#\text{ equations}$$

$$2 = 4 - 2.$$

This means the solution will have the form

$$\vec{x} = \vec{p} + s\vec{u} + t\vec{v}.$$

two free parameters.

How do we find $\vec{p}, \vec{u}, \vec{v}$?

No short cut this time (no cross product in $\mathbb{R}^4$), so we have to use elimination:

Try to eliminate x_1:

$$\begin{array}{rl} (1) & x_1 + 3x_2 + 0 + 2x_4 = 1 \\ -(2) & x_1 + 3x_2 + x_3 + 6x_4 = 7 \\ \hline (3) & 0 + 0 - x_3 - 4x_4 = -6 \end{array}$$

$$x_3 + 4x_4 = 6.$$

OOPS, we also eliminated x_2!

That's OK

So we get a new, equivalent system that is simpler:

$$\begin{cases} ① & x_1 + 3x_2 + 0 + 2x_4 = 1, \\ ③ & 0 + 0 + x_3 + 4x_4 = 6. \end{cases}$$

I claim that we are done; we cannot simplify any further.

Let me show you how to read off the solution:

- variables in corners of the staircase are called

  PIVOT variables: x_1 & x_3.

- all other variables are called

  FREE variables: x_2 & x_4.

Usually we like to rename the free variables; let's say

$s = x_2$ & $t = x_4$.

Now we just write down the solution:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1-3s-2t \\ s \\ 6-4t \\ t \end{pmatrix}$$

$$= \begin{pmatrix} 1-3s-2t \\ 0+1s+0t \\ 6+0s-4t \\ 0+0s+1t \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \\ 6 \\ 0 \end{pmatrix} + s\begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -2 \\ 0 \\ -4 \\ 1 \end{pmatrix}$$

$$= \vec{p} + s\vec{u} + t\vec{v}.$$

A parametrized 2-plane in $\mathbb{R}^4$

[I will say much more about pivot & free variables next time....]