HW2 due <u>now</u>.

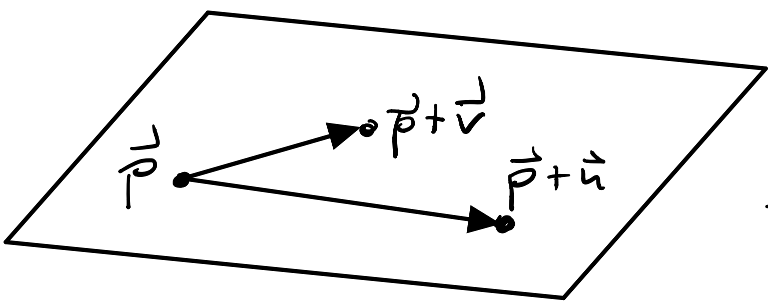
Solutions up by <u>tomorrow</u>.

Quiz 2 at beginning of Tuesday's class.

Today: HW2 Discussion.

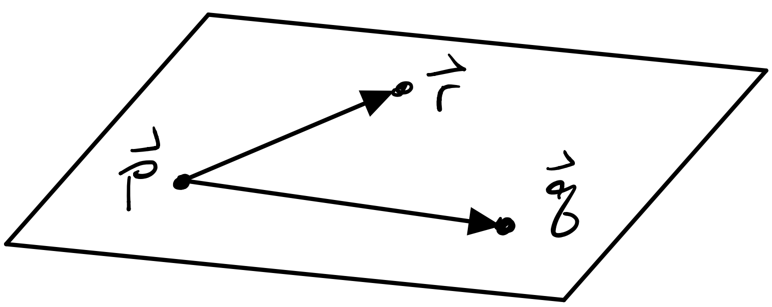
<u>Problem 1</u>: 3 basic ways to describe a plane in $\mathbb{R}^3$:

- point & 2 direction vectors.



$$\vec{x} = \vec{p} + s\vec{u} + t\vec{v}$$

- 3 points

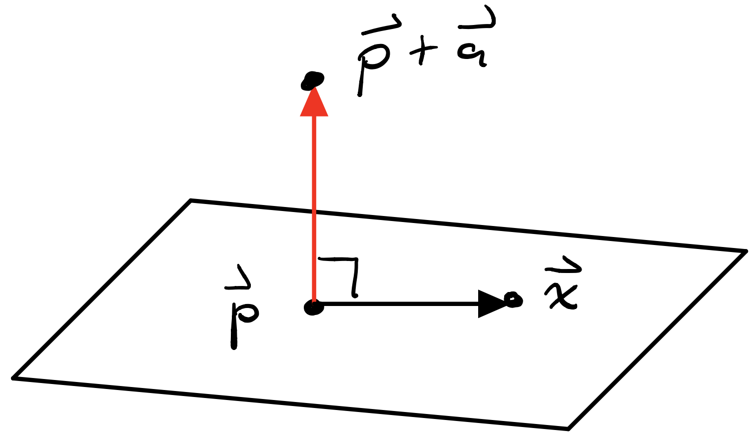


take

$\vec{u} = \vec{q} - \vec{p}$

$\vec{v} = \vec{r} - \vec{p}$

- point & normal vector



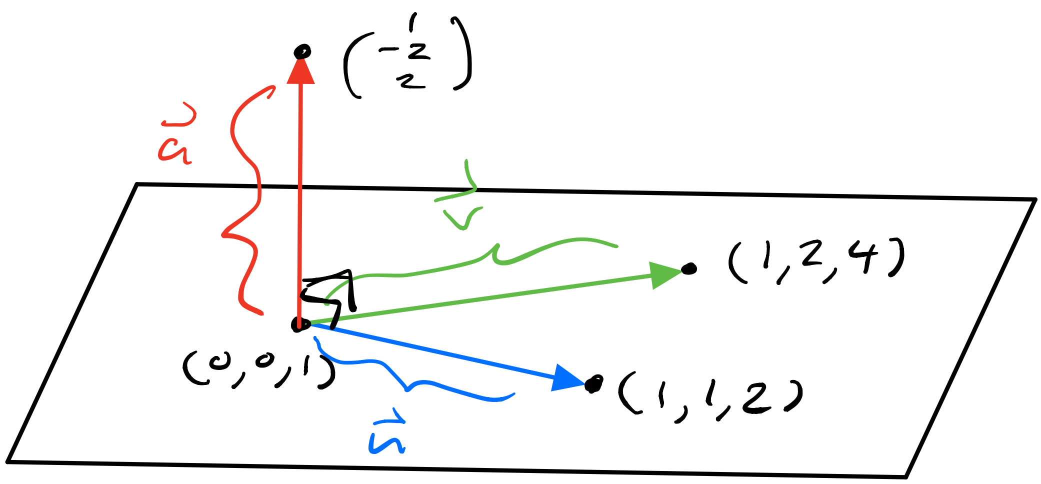
$$\vec{a}\cdot(\vec{x}-\vec{p}) = 0$$
$$\vec{a}\cdot\vec{x} - \vec{a}\cdot\vec{p} = 0$$
$$\vec{a}\cdot\vec{x} = \vec{a}\cdot\vec{p}$$

To convert between various descriptions:

1(a): Given $\vec{x} = \vec{p} + s\vec{u} + t\vec{v}$, convert to the form $\vec{a}\cdot\vec{x} = \vec{a}\cdot\vec{p}$.

Many ways to do this.

Quickest way: Take $\vec{a} = \vec{u}\times\vec{v}$.



To find $\vec{a} = (a,b,c)$ that is $\perp$ to both $\vec{u} = (1,1,1)$ & $\vec{v} = (1,2,3)$, we take the cross product:

$$\vec{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3-2 \\ 1-3 \\ 2-1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

Done. The equation is

$$\vec{a} \circ \vec{x} = \vec{a} \circ \vec{p}$$

$$(1,-2,1) \circ (x,y,z) = (1,-2,1) \circ (0,0,1)$$

$$x - 2y + z = 1.$$

///

[There are other ways to get the same answer, this is just the quickest.]

1(b): Given $ax + by + cz = d$, convert to the form $\vec{x} = \vec{p} + s\vec{u} + t\vec{v}$.

Quickest Way: Take $s = y$ & $t = z$.

Then $ax + by + cz = d$

$$\Rightarrow \quad x = \frac{d}{a} - \frac{b}{a}y - \frac{c}{a}z \qquad (\text{assume } a \neq 0)$$

and hence $x = \frac{d}{a} - \frac{b}{a}s - \frac{c}{a}t$.

Example: $x + 2y + 4z = 6$

Let $s = y$ & $t = z$, so

$x = 6 - 2y - 4z = 6 - 2s - 4t$, hence

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 - 2s - 4t \\ 0 + 1s + 0t \\ 0 + 0s + 1t \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}}_{\vec{p}} + s \underbrace{\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}}_{\vec{u}} + t \underbrace{\begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}}_{\vec{v}}$$

a point on the plane

two direction vectors in the plane.

[Remark: 1(b) was easier than 1(a).]

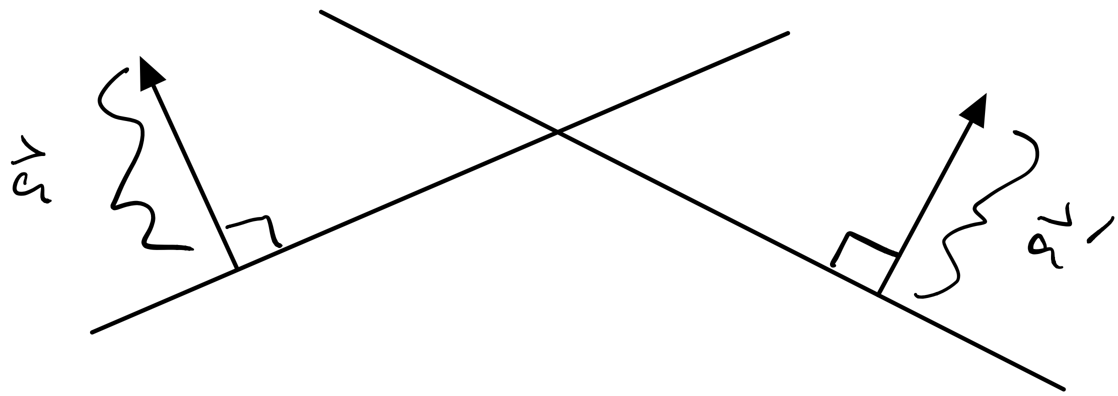
Problem 2:

Consider two lines

$$\vec{a} \cdot \vec{x} = c \quad \& \quad \vec{a}' \cdot \vec{x} = c'$$

$$ax + by = c \quad \& \quad a'x + b'y = c'$$

Picture:



(values of c & c' not shown)

Observe that lines are perpendicular or parallel if and only if the corresponding normal vectors $\vec{a}$ & $\vec{a}'$ are perpendicular or parallel.

2(a): They are <u>perpendicular</u>

$$\Longleftrightarrow \quad \vec{a} \cdot \vec{a}' = 0$$

$$(a,b)\cdot(a',b')=0$$

$$\boxed{aa'+bb'=0}$$

Alternatively, the slopes are $-a/b$ & $-a'/b'$. Perpendicular means "negative reciprocal slopes":

$$-\frac{a}{b}=+\frac{b'}{a'}$$

$$-aa'=bb'$$

$$aa'+bb'=0 \quad \checkmark$$

2(b): The vectors are parallel when

$\vec{a}'=t\vec{a}$ for some t.

$$(a',b')=t(a,b)$$

$$\begin{Bmatrix} a'=ta \\ b'=tb \end{Bmatrix}$$

$$\frac{a'}{a}=t=\frac{b'}{b}.$$

I like to simplify as follows:

$$\frac{a'}{a} = \frac{b'}{b}$$

$$a'b = ab'$$

$$\boxed{ab' - a'b = 0.}$$

Alternatively, the slopes are $-a/b$ & $-a'/b'$. Parallel means equal slope:

$$-\frac{a}{b} = -\frac{a'}{b'}$$

$$-ab' = -a'b$$

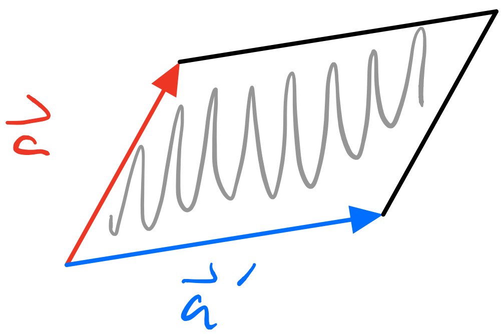
$$ab' - a'b = 0 \quad \checkmark$$

JARGON: Given two vectors (a,b) & (a',b') we define the determinant as follows:

$$\det\begin{pmatrix} a & a' \\ b & b' \end{pmatrix} = \underline{ab'} - \underline{a'b}$$

we just showed that det = 0 if & only if the vectors are <u>parallel</u>.

Geometry: det is $\pm$ area of the parallelogram:



$$\text{area} = \pm\det\left(\vec{a}\ \ \vec{a}\,'\right)$$

[Proof: Never mind...]

Problem 4: Cross Product.

Given $\vec{u} = (u, v, w)$ & $\vec{u}\,' = (u', v', w')$,
we define the <u>cross product vector</u>

$$\vec{u} \times \vec{u}' = (vw' - v'w,\ u'w - uw',\ uv' - u'v).$$

WHY?!!!

4(a): Because the cross product is simultaneously $\perp$ to both $\vec{u}$ & $\vec{u}'$?

Check. Let $\vec{a} = \vec{u} \times u'$. Then

$$\begin{aligned}\vec{u} \cdot \vec{a} &= u(vw' - v'w) \\ &\quad + v(u'w - uw') \\ &\quad + w(uv' - u'v)\end{aligned}$$

$$\begin{aligned}&= \cancel{uvw'} - \cancel{uv'w} \\ &\quad + \cancel{u'vw} - \cancel{uvw'} \\ &\quad + \cancel{uv'w} - \cancel{u'vw} = 0.\end{aligned}$$

So it works. ✓

[See solutions for $\vec{u}' \cdot \vec{a} = 0$]

4(b): Application. Solve

$$\begin{cases} x + y + 2z = 0, \\ 3x + 4y + 5z = 0. \end{cases}$$

In other words:

$$\begin{pmatrix}1\\1\\2\end{pmatrix} \cdot \begin{pmatrix}x\\y\\z\end{pmatrix} = 0,$$

$$\begin{pmatrix}3\\4\\5\end{pmatrix} \cdot \begin{pmatrix}x\\y\\z\end{pmatrix} = 0.$$

Need a vector $\vec{x}$ that is simultaneously $\perp$ to $(1,1,2)$ & $(3,4,5)$.

Take

$$\vec{x} = \begin{pmatrix}1\\1\\2\end{pmatrix} \times \begin{pmatrix}3\\4\\5\end{pmatrix} = \begin{pmatrix}5-8\\6-5\\4-3\end{pmatrix} = \begin{pmatrix}-3\\1\\1\end{pmatrix}.$$

That's just <u>one</u> solution. The full solution is a <u>line</u>:

$$\vec{x} = t\begin{pmatrix}-3\\1\\1\end{pmatrix}.$$

[still $\perp$ to both $(1,1,2)$ & $(3,4,5)$].

Given 3 vectors in $\mathbb{R}^3$:

$\vec{a} = (a, b, c)$

$\vec{a}' = (a', b', c')$

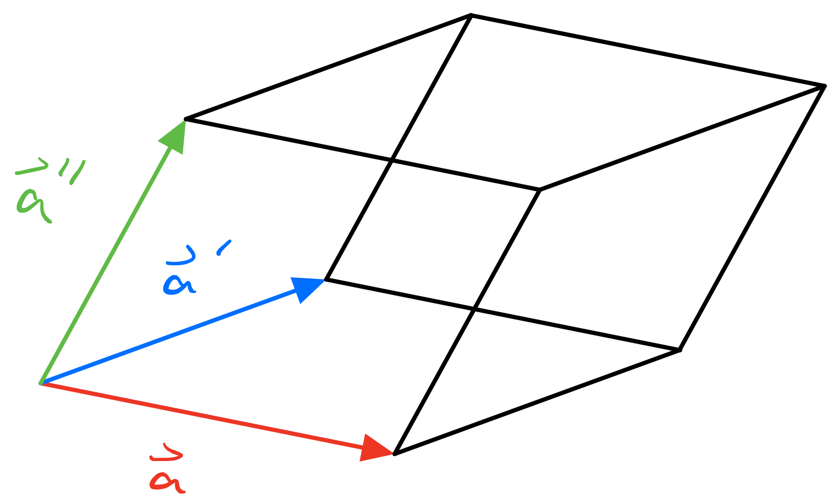
$\vec{a}'' = (a'', b'', c'')$,

we define the <u>determinant</u>

$$\det\begin{pmatrix} \vec{a} & \vec{a}' & \vec{a}'' \end{pmatrix} := \vec{a} \cdot (\vec{a}' \times \vec{a}'').$$

WHY ?!!! Secret reason:

det = $\pm$ volume of the parallepiped (squashed box) generated by $\vec{a}, \vec{a}', \vec{a}''$.



$$\text{volume} = \pm \det\begin{pmatrix} | & | & | \\ \vec{a} & \vec{a}' & \vec{a}'' \\ | & | & | \end{pmatrix}$$

Proof: Omitted :)

Application: $\det = 0$

$\Longleftrightarrow$ the vectors $\vec{a}, \vec{a}', \vec{a}''$ all live in the same plane.

Closing Remark: Determinants are tricky but quite useful. They will go away for a while and then show up near the end of the course.

Problem 5:

$$\begin{aligned} &① \\ &② \\ &③ \end{aligned} \quad \begin{cases} x + y + 2z = 0, \\ 3x + 4y + 5z = 0, \\ x + 2y + cz = -2. \end{cases}$$

Intersection of 3 planes in $\mathbb{R}^3$.

Planes ① & ② intersect in the line

$$\vec{x} = t(-3,1,1)$$

$$(x,y,z) = (-3t, t, t)$$

To find the point of intersection of ①, ② & ③, we substitute these values of x, y, z into ③:

$$x + 2y + cz = -2$$

$$(-3t) + 2(t) + c(t) = -2$$

$$(c-1)t = -2.$$

5(a): When $c = 4$ we get

$$(4-1)t = -2$$

$$3t = -2$$

$$t = -2/3,$$

hence the 3 planes intersect at the point

$$(x,y,z) = -\frac{2}{3}(-3,1,1) = \left(2, \frac{-2}{3}, \frac{-2}{3}\right).$$

5(b): When $c = 1$, then we get

$$(c-1)t = -2$$
$$0t = -2,$$

which has NO SOLUTION. So in this case, the 3 planes have no common point of intersection.

But we also observe that no 2 of the planes are parallel:

$$\begin{cases} x + y + 2z = 0 \\ 3x + 4y + 5z = 0 \\ x + 2y + z = -2 \end{cases}$$

No 2 of the vectors $(1,1,1), (3,4,5), (1,2,1)$ are parallel.

Picture: