Quiz 1 : Average 8.09/10.

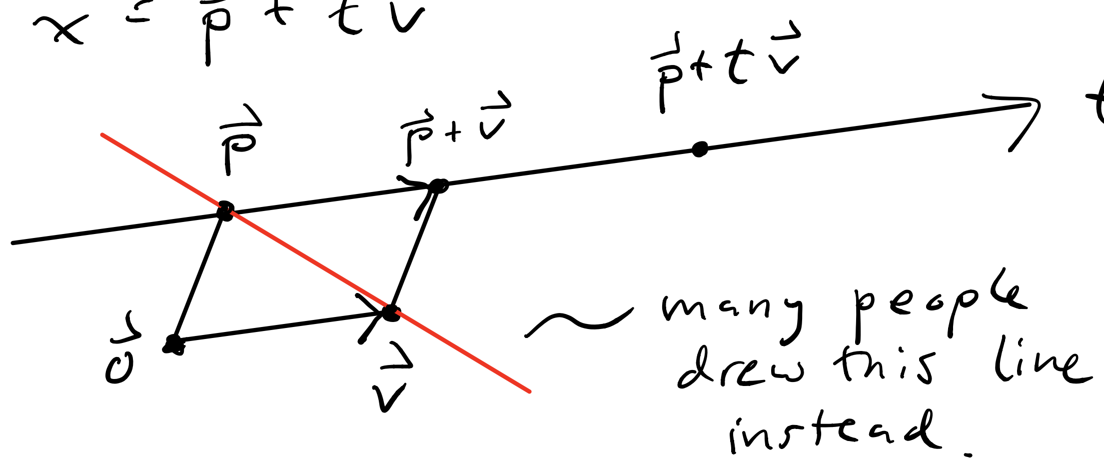
HW2 due Thursday <u>before</u> class.

office Hours tomorrow 4pm.

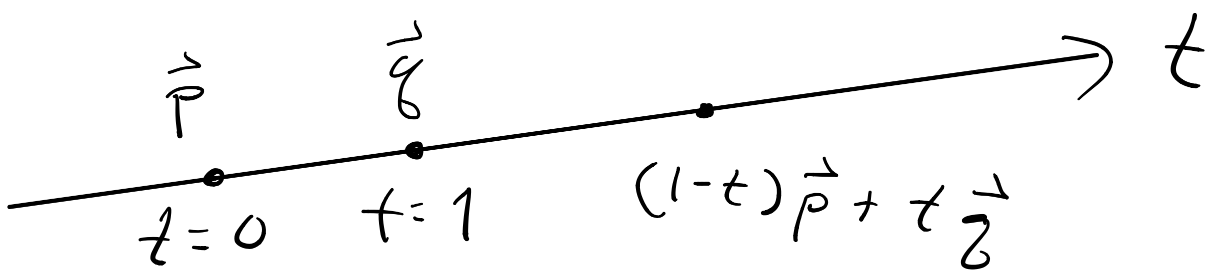
Quiz 2 begin <u>Tuesday</u>'s class.

Quiz 1 Problem 1 review :

(a) $\vec{x} = \vec{p} + t\vec{v}$

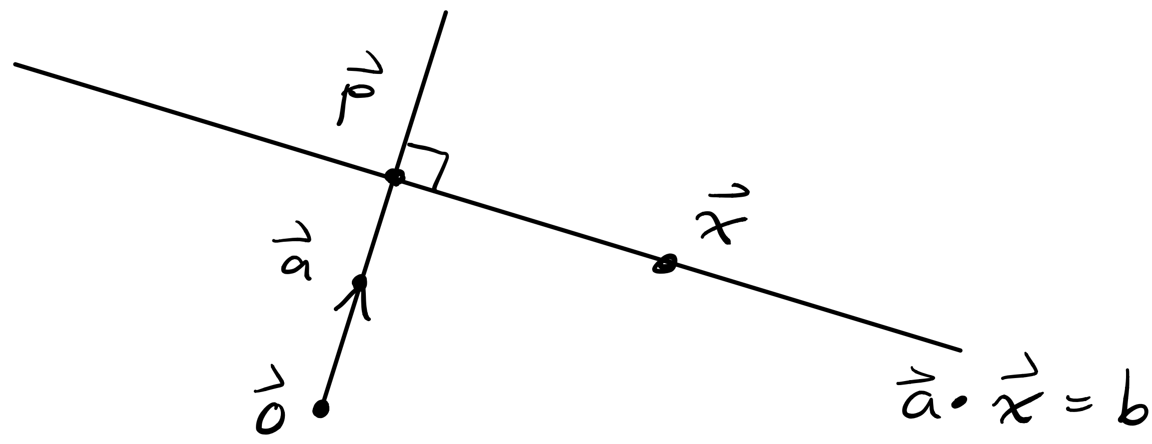


(b) $\vec{x} = (1-t)\vec{p} + t\vec{q}$



(c) $\vec{a} \cdot \vec{x} = b$

This is a line that is $\perp$ to the vector $\vec{a}$:



The equation $\vec{a} \cdot \vec{x} = b$ does not explicitly tell us a point on the line, but we can find one.

Let $\vec{p}$ be intersection of

$$\vec{x} = t\vec{a} \quad \& \quad \vec{a} \cdot \vec{x} = b.$$

Solve for $\vec{p}$.

Solution: We know 2 things.

- $\vec{p} = t\vec{a}$ for some t.
- $\vec{a} \cdot \vec{p} = b$.

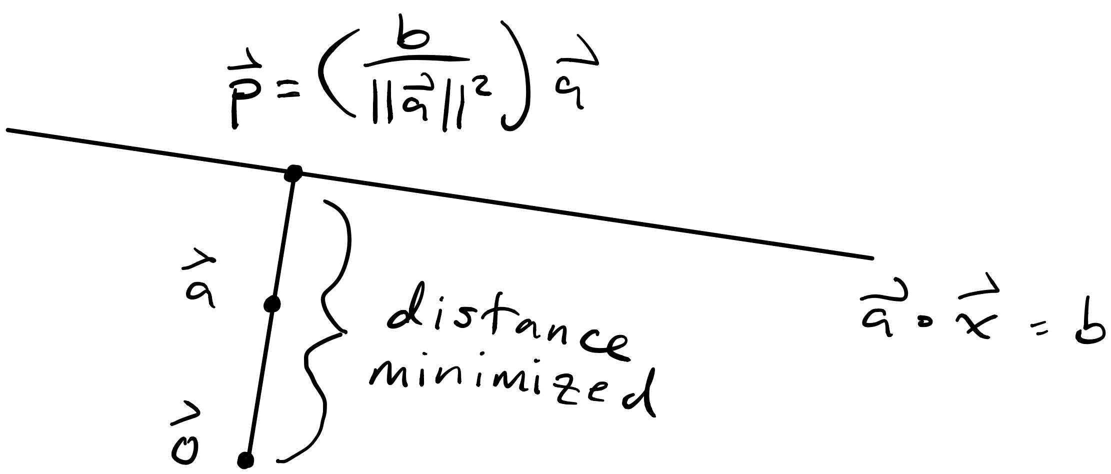
Given $\vec{a}$ & b, want t.

Substitute:

$$\vec{a} \cdot (t\vec{a}) = b$$
$$t(\vec{a} \cdot \vec{a}) = b$$
$$t\|\vec{a}\|^2 = b$$
$$t = b/\|\vec{a}\|^2.$$

Conclusion: The point $\vec{p} = \left(\frac{b}{\|\vec{a}\|^2}\right)\vec{a}$ is a point on the line $\vec{a} \cdot \vec{x} = b$.

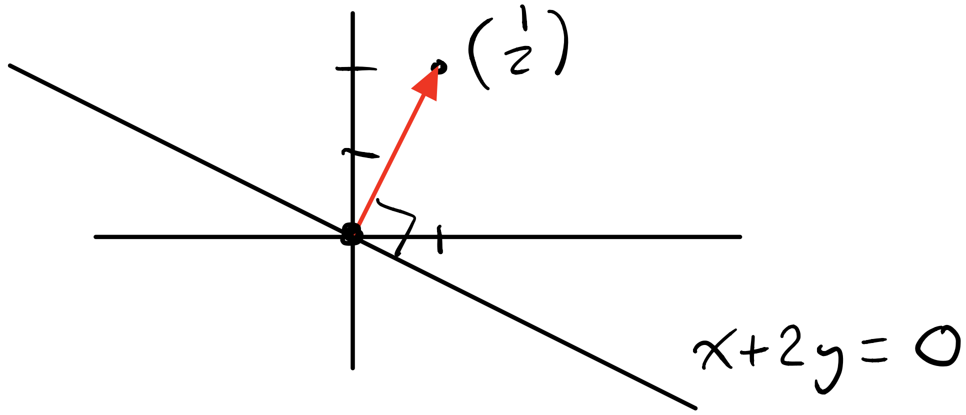
In fact it is the closest point on this line to the origin.



Special Case: When $b = 0$, this line goes through the origin.

e.g. $(1,2)\cdot(x,y) = 0$

is $\perp$ to vector $(1,2)$ & contains the origin $(0,0)$:



Next Topic:

A system of m linear equations in n unkowns looks like:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

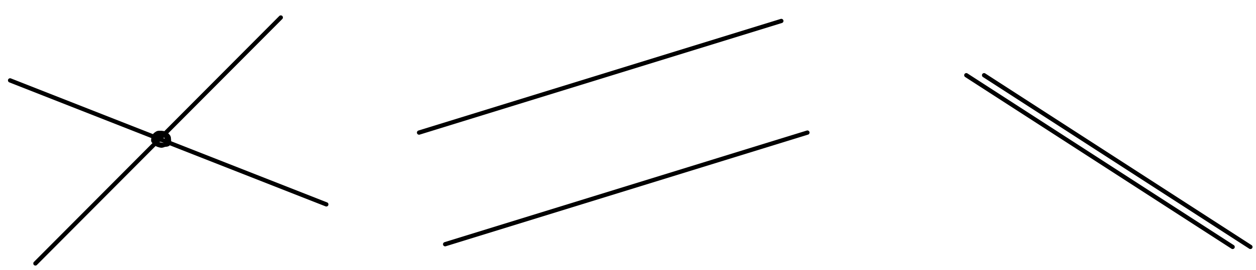
Each such equation defines a hyperplane ($(n-1)$-dim) living in $\mathbb{R}^n$.

The system defines the <u>intersection</u> of m such hyperplanes.

Humans completely understand this problem, soon we will too.

Small Examples:

- $m = 2$, $n = 2$: 2 lines in $\mathbb{R}^2$



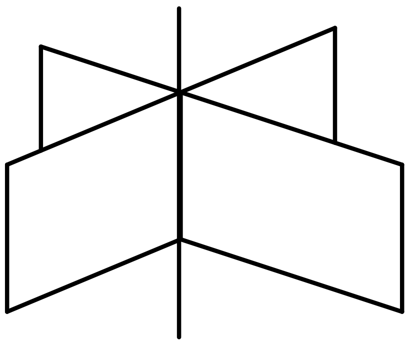
a point   empty   a line

Algebraically:

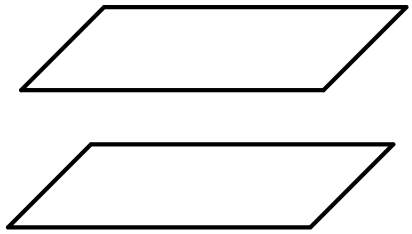
$$\begin{cases} ax + by = c, \\ a'x + b'y = c'. \end{cases}$$

See HW 2.1 for discussion.

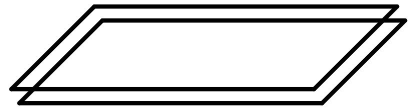
- $m=2,\ n=3$: 2 planes in $\mathbb{R}^3$.



a line



empty



a plane

Algebraically:

$$\begin{cases} ax+by+cz=d \\ a'x+b'y+c'z=d' \end{cases}$$

Let $\vec{a}=(a,b,c)$ & $\vec{a}'=(a',b',c')$.

We define the <u>cross product</u>

$$\vec{v}=\vec{a}\times\vec{a}'=(bc'-b'c,\ a'c-ac',\ ab'-a'b).$$

If $\vec{v}\neq(0,0,0)$ then the planes intersect in a line $\vec{x}=\vec{p}+t\vec{v}$ for some point $\vec{p}$ (easy to find).

If $\vec{v}=(0,0,0)$ then the two

planes are <u>parallel</u>.

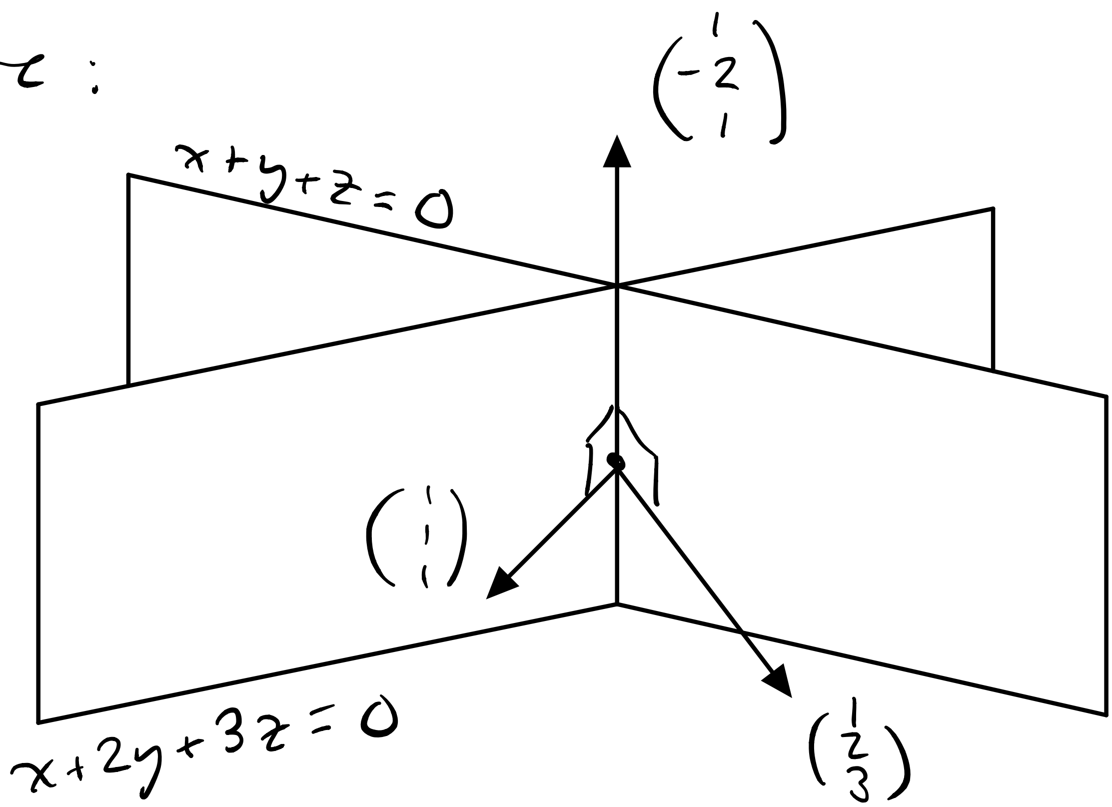
[I'll explain more later. It has to do with the concept of a "determinant."]

Recall: $\begin{cases} x+y+z = 0, \\ x+2y+3z = 0. \end{cases}$

Cross Product of the normal vectors

$$\begin{pmatrix}1\\1\\1\end{pmatrix} \times \begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}3-2\\1-3\\2-1\end{pmatrix} = \begin{pmatrix}1\\-2\\1\end{pmatrix}.$$

Picture:



vector $\begin{pmatrix}1\\1\\1\end{pmatrix}$ is $\perp$ plane $x+y+z=0$

vector $\begin{pmatrix}1\\2\\3\end{pmatrix}$ is $\perp$ plane $x+2y+3z=0$

vector $\begin{pmatrix}1\\-2\\1\end{pmatrix}$ is $\perp$ $\begin{pmatrix}1\\1\\1\end{pmatrix}$ & $\begin{pmatrix}1\\2\\3\end{pmatrix}$,

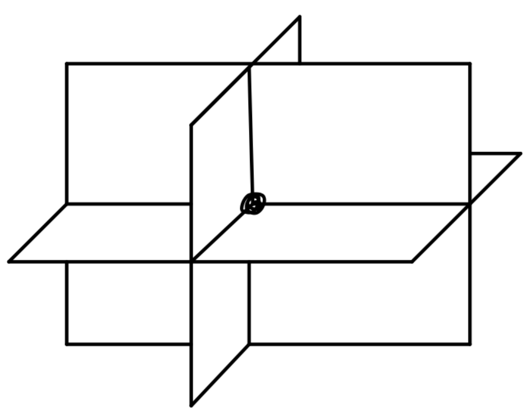
hence

vector $\begin{pmatrix}1\\-2\\1\end{pmatrix}$ is $\parallel$ to both planes.

[See HW 2 Problem 4.]

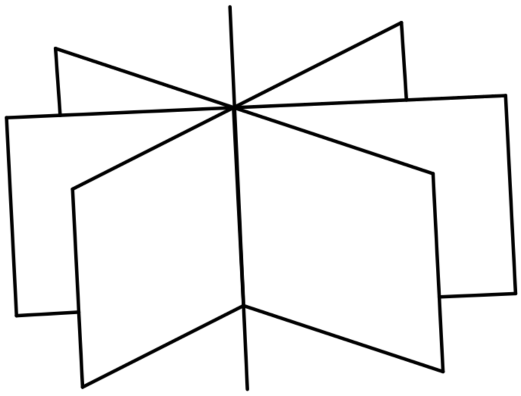
- $m=3$, $n=3$:

Intersection of 3 planes in $\mathbb{R}^3$.
What could happen?

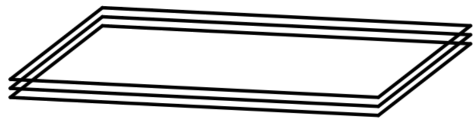


<u>Most likely</u>:
intersect at a single point.

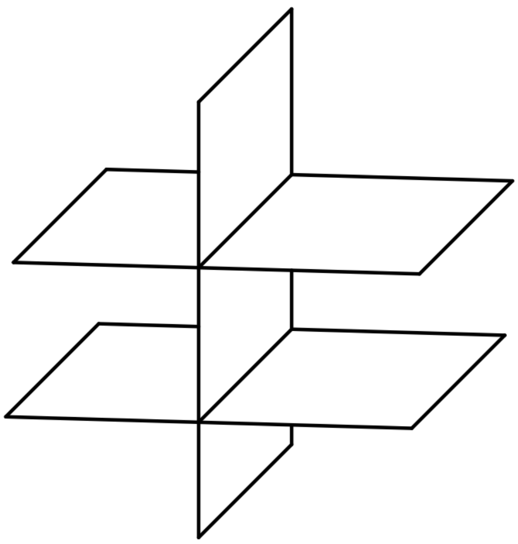
Some other possibilities:

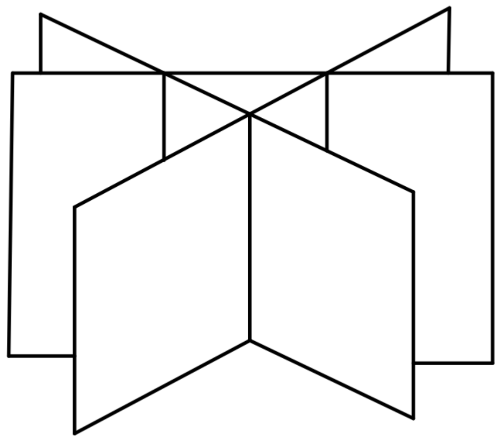


a line



a plane





NO SOLUTIONS because the planes are parallel in some way.

Algebraically:

Solve for (x,y,z) in

$$\begin{cases} ax + by + cz = d, \\ a'x + b'y + c'z = d', \\ a''x + b''y + c''z = d''. \end{cases}$$

How to connect the algebra & geometry?

Let's add a 3rd plane to our previous example:

$$\begin{array}{c} (1) \\ (2) \\ (3) \end{array} \begin{cases} x+y+z=0 \\ x+2y+3z=0 \\ x+2y+cz=1 \end{cases}$$

where c is some constant.

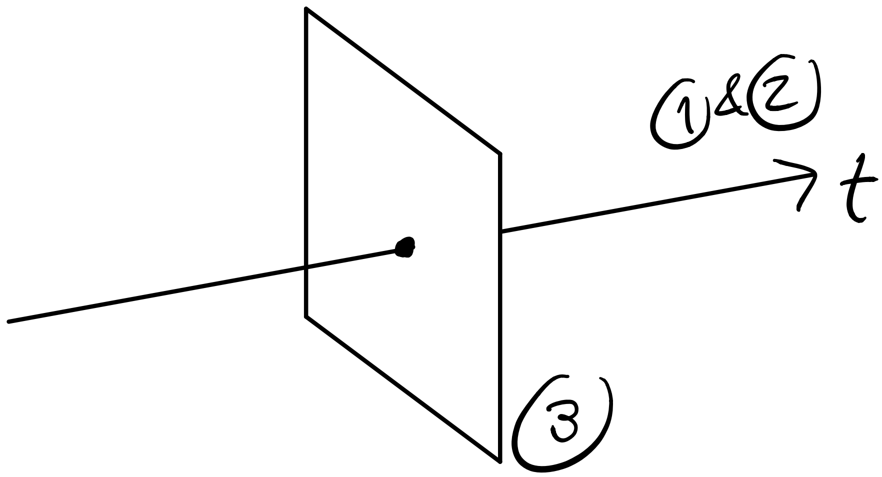
Solve for (x,y,z).

We already know that (1) & (2) intersect in the line

$$(x,y,z) = t(1,-2,1) = (t,-2t,t)$$

Now:

intersection of planes (1), (2), (3) = intersection of line (1) & (2) with plane (3).



Solve for t to get the point of intersection $(x,y,z) = (t,-2t,t)$.

Plug into 3:

$$x + 2y + cz = 1$$

$$(t) + 2(-2t) + c(t) = 1$$

$$\boxed{(c-3)\,t = 1}$$

There are two cases:

- If $c \neq 3$, then $t = 1/(c-3)$ and the solution is a unique point:

$$(x,y,z) = t(1,-2,1) = \frac{1}{c-3}(1,-2,1).$$

DONE.

- If $c = 3$ then the equation

$$(c-3)t = 1$$
$$0t = 1$$

has NO SOLUTION, and therefore the original system has NO SOLUTION.

What is the picture?

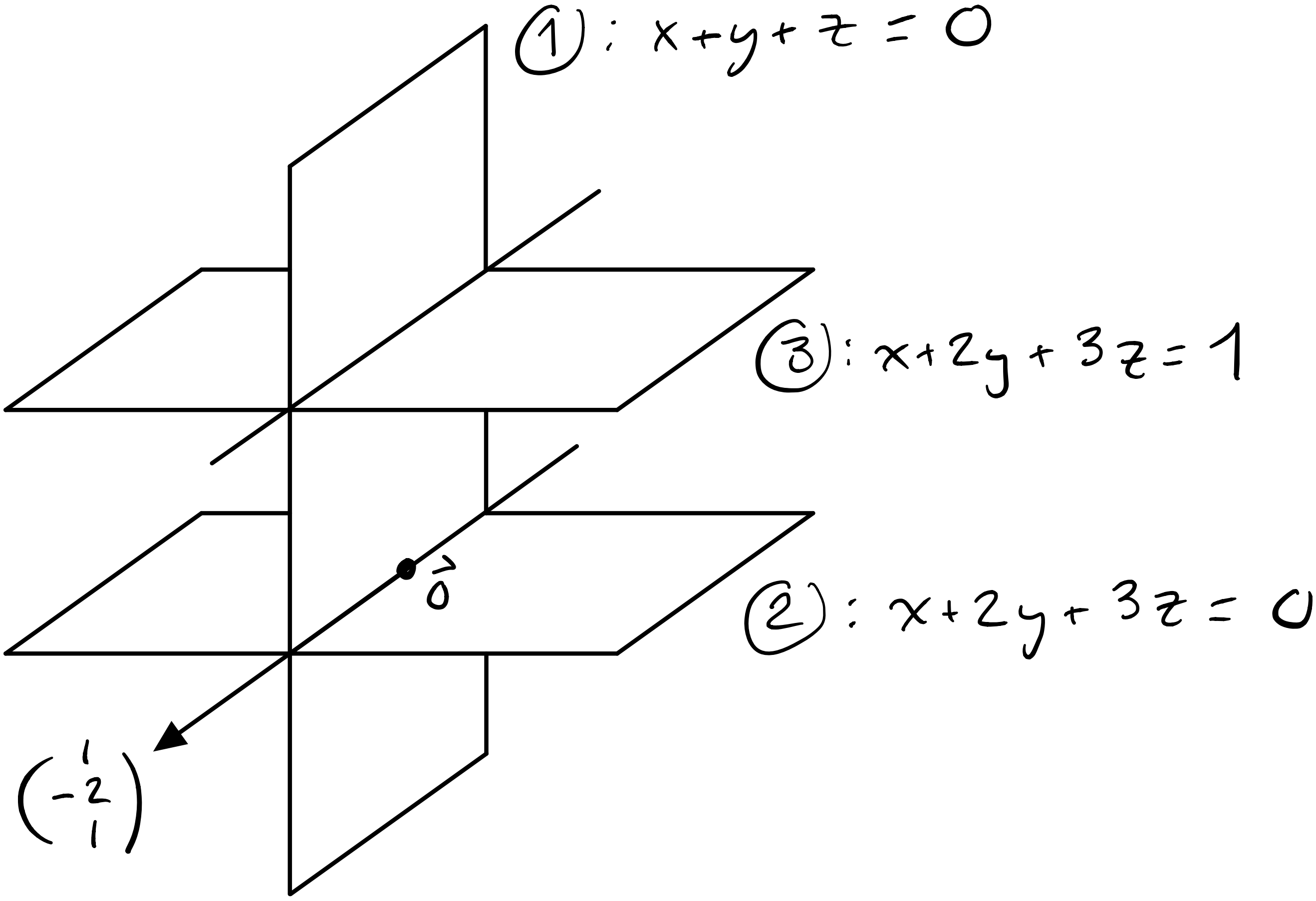
$$\begin{cases} x + y + z = 0, & (1) \\ x + 2y + 3z = 0, & (2) \\ x + 2y + 3z = 1. & (3) \end{cases}$$

Recall: The planes $\vec{a} \cdot \vec{x} = b$
$\vec{a} \cdot \vec{x} = b'$

are parallel to each other (and both $\perp$ to the vector $\vec{a}$).

In our case planes (2) & (3) are parallel (and both $\perp$ to vector $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$).

Here is a picture:



Summary:

- ① & ③ meet in a line.

  [Exercise: Describe this line.]

- ① & ② meet in the line $t\begin{pmatrix}1\\-2\\1\end{pmatrix}$.

- But ② & ③ are parallel, so they don't meet at all.

See HW2 Problem 5.