HW2 due Thurs Sept 17, <u>before the lecture</u>.

Topic: Planes in $\mathbb{R}^3$.

A line in n-dimensional space $\mathbb{R}^n$ has the form $\vec{x} = \vec{p} + t\vec{v}$, for some point $\vec{p}$ & direction vector $\vec{v}$.

$$(x_1, \dots, x_n) = (p_1 + tv_1, \dots, p_n + tv_n).$$

In $\mathbb{R}^2$, we can also define a line by an equation of the form

$$ax + by = c,$$

but this does not work in higher dimensions.

In $\mathbb{R}^3$: $ax + by + cz = d$ does not define a <u>line</u>, instead it defines a <u>plane</u>.

In $\mathbb{R}^4$: $ax + by + cz + dw = e$ defines a "3-dimensional plane" living in "4-dimensional space."

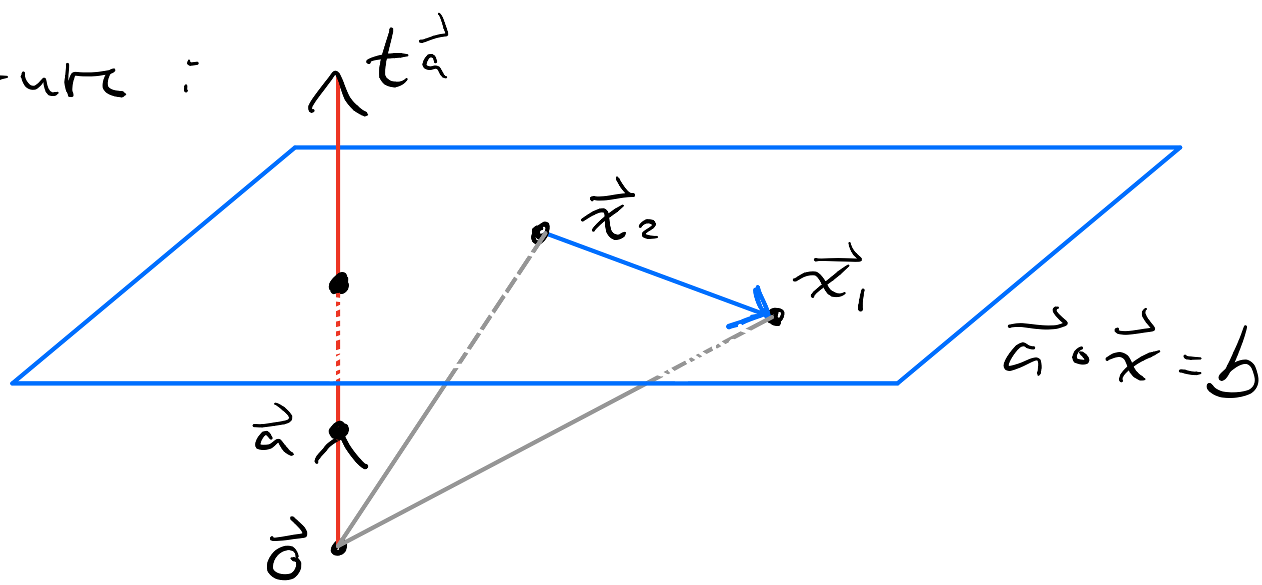
In $\mathbb{R}^n$: One linear equation in n unknowns,

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b,$$

defines an "$(n-1)$-dimensional plane" living in $\mathbb{R}^n$.

Jargon: We call it a "hyperplane."

Picture:



Given a normal vector $\vec{a} = (a_1, a_2, \cdots, a_n) \in \mathbb{R}^n$, the set defined by $\vec{a} \cdot \vec{x} = b$ is a hyperplane $\perp$ to line $t\vec{a}$.

Indeed, for any two points $\vec{x}_1$ & $\vec{x}_2$ in the hyperplane we have

$$\begin{cases} \vec{a} \cdot \vec{x}_1 = b \\ \vec{a} \cdot \vec{x}_2 = b \end{cases} \leftarrow \text{points } \vec{x}_1 \text{ \& } \vec{x}_2 \text{ are both in the hyperplane}$$

hence $\vec{a} \cdot (\vec{x}_1 - \vec{x}_2)$

$$= \vec{a} \cdot \vec{x}_1 - \vec{a} \cdot \vec{x}_2 = b - b = 0.$$

Idea: One linear equation in n unknowns is a hyperplane in $\mathbb{R}^n$.

And m simultaneous linear eqns in n unknowns is the intersection of m hyperplanes in $\mathbb{R}^n$.

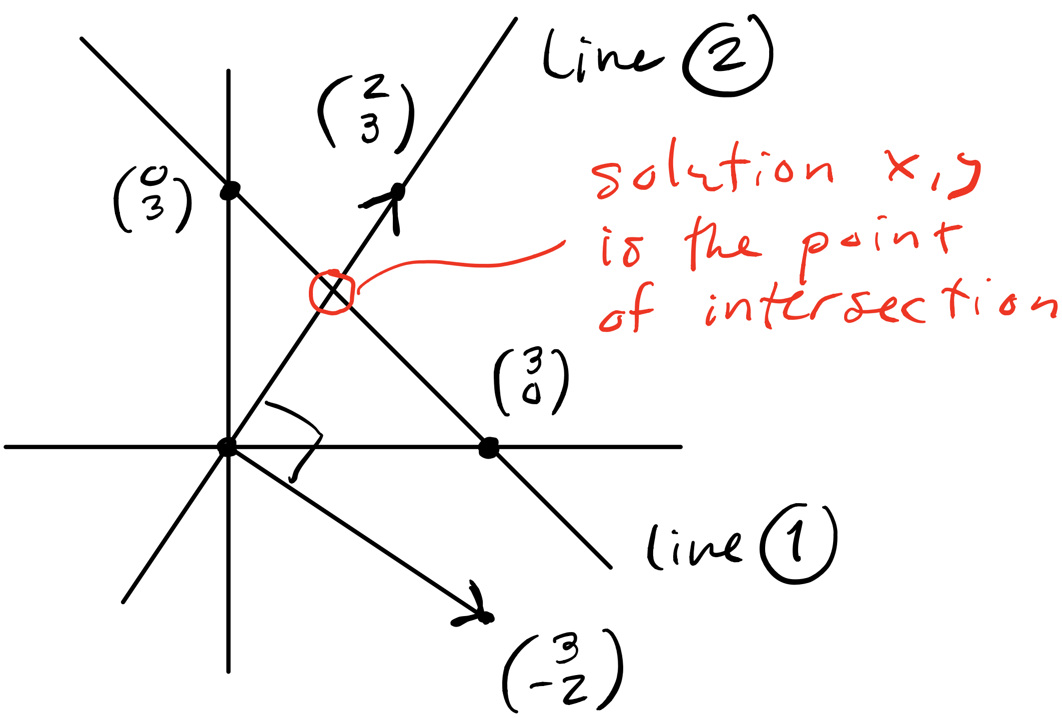
Apologies if this doesn't make sense right away.

Some small examples:

Solve the following system of 2 linear equations in 2 unknowns:

$$\begin{cases} x + y = 3 & (1) \\ 3x - 2y = 0 & (2) \end{cases}, \quad \left(\begin{array}{l} \vec{a} \cdot \vec{x} = 0 \\ \text{is} \perp \text{vector } \vec{a} \end{array} \right)$$

Represents the intersection of two lines in $\mathbb{R}^2$.



Two methods:

- Express line 2 as

$$(x,y) = t(2,3) = (2t, 3t)$$

Then substitute into line 1 to get

$$x + y = 3$$
$$(2t) + (3t) = 3$$
$$5t = 3$$
$$t = 3/5.$$

Therefore the point of intersection is $(x,y) = t(2,3) = \frac{3}{5}(2,3)$

$$= \left(\frac{6}{5}, \frac{9}{5}\right).$$

- The method of "elimination":

Combine equations 1 & 2 to get a new equation that does not contain x. So let's define

$$③ = ② - 3①.$$

$$\begin{array}{rrcrcr} ② & 3x & - & 2y & = & 0 \\ -3① & -3x & - & 3y & = & -9 \\ \hline ③ & 0x & - & 5y & = & -9 \end{array}$$

$$-5y = -9$$

$$y = 9/5.$$

Plug this into either equation to get x. Eg, plug into ①:

$$x + y = 3$$

$$x + \frac{9}{5} = 3$$

$$x = 3 - \frac{9}{5} = \frac{6}{5} \checkmark$$

SAME ANSWER $\checkmark$

What if the lines are parallel?

Solve
$$\begin{cases} x + 2y = 1 & (1) \\ x + 2y = 0 & (2) \end{cases}$$

Try eliminating x:

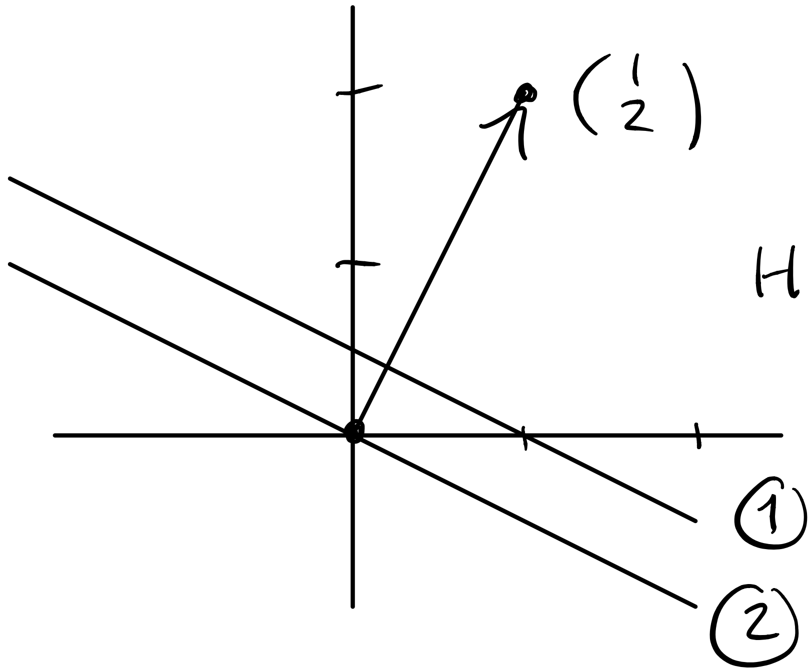
$$\begin{array}{rrcl} (1) & x + 2y &=& 1 \\ -(2) & -x - 2y &=& -0 \\ \hline (3) & 0x + 0y &=& 1 \end{array} \quad (0 = 1)$$

Observe: (3) has NO SOLUTION!

If (1) & (2) have a solution,
then (3) has a solution.

But (3) does not have a solution,
therefore (1) & (2) has no solution.

Picture :



HW 1.5(a)

The lines do not intersect because they are PARALLEL !

[See Problem 3 on HW2.]

How about 3 unknowns ?

Solve
$$\begin{cases} x + y + z = 0, & (1) \\ x + 2y + 3z = 0. & (2) \end{cases}$$

What do you expect ?

The intersection of two planes in $\mathbb{R}^3$ is .... <u>a line</u> (probably).

Let's find the line.

Try to eliminate x:

$$
\begin{array}{rrcrcrcl}
(2) & x & + & 2y & + & 3z & = & 0 \\
-(1) & -x & - & y & - & z & = & 0 \\
\hline
(3) & 0x & + & 1y & + & 2z & = & 0
\end{array}
$$

$$y + 2z = 0.$$

Great, now what?

When are we supposed to stop?

Let's try to eliminate y instead:

$$
\begin{array}{rrcrcrcl}
(2) & x & + & 2y & + & 3z & = & 0 \\
-2(1) & -2x & - & 2y & - & 2z & = & 0 \\
\hline
(4) & -x & + & 0 & + & z & = & 0
\end{array}
$$

<u>or</u> $x + 0 - z = 0$

Key Fact: The system 1 & 2 has the same solutions as the system 3 & 4, which is simpler.

$$\begin{Bmatrix} x+y+z=0 \\ x+2y+3z=0 \end{Bmatrix} \Longleftrightarrow \begin{Bmatrix} x \quad -z=0 \\ y+2z=0 \end{Bmatrix}$$

EQUIVALENT SYSTEMS.

In fact, we can easily see that the solution to 3 & 4 is a line.

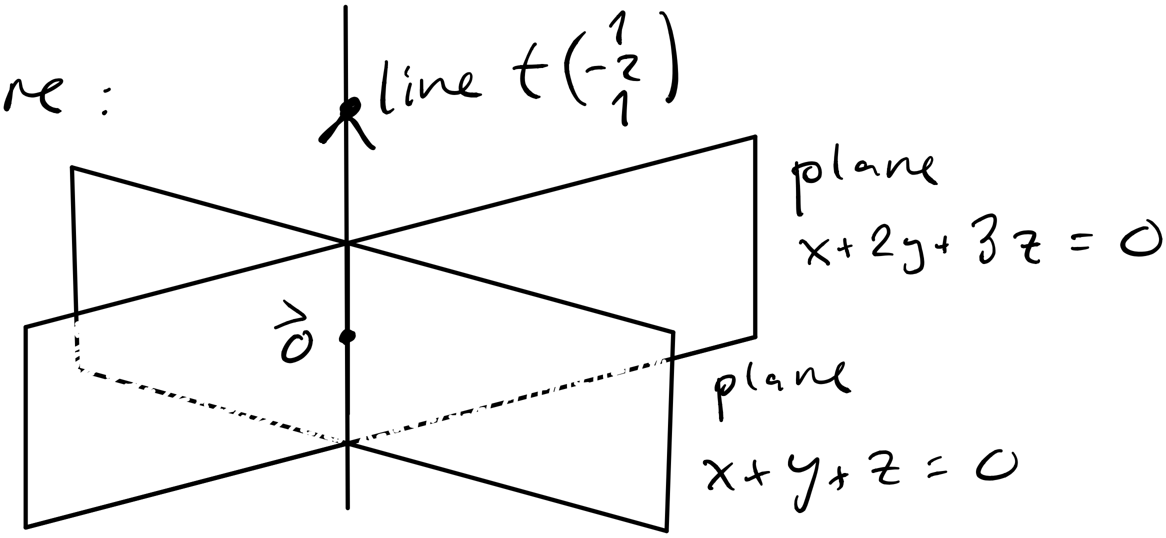
$$\begin{Bmatrix} x \quad -z=0 \\ y+2z=0 \end{Bmatrix} \Longleftrightarrow \begin{Bmatrix} x=z \\ y=-2z \end{Bmatrix}$$

Solution:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ -2z \\ z \end{pmatrix} = z \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

The line in direction $(1,-2,1)$.

Picture:



[We couldn't eliminate all 3 variables because we didn't start with 3 equations. If we start with 2 equations in 3 unknowns, then we will probably have 1 variable left over; in this case z.]

Idea of "Dimension":

Given m equations in n ($\geq m$) unknowns, we can eliminate m unknows, with $n-m$ left over. In this case the solution will be "$(n-m)$-dimensional."

A Shortcut:

To find the intersection of 2 planes in $\mathbb{R}^3$,

$$\begin{cases} ax + by + cz = 0, \\ a'x + b'y + c'z = 0. \end{cases}$$

I claim that the solution is the line $\vec{x} = t\vec{v}$, where the vector $\vec{v}$ is defined by

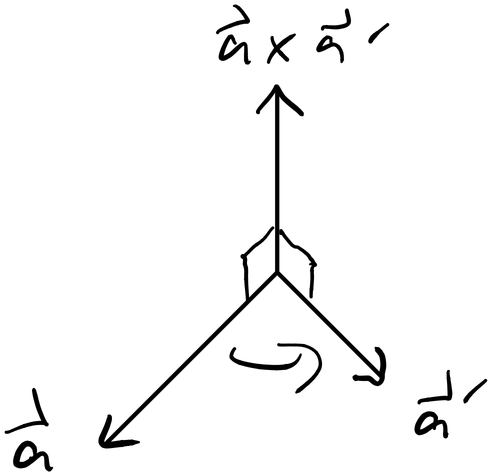
$$\vec{v} = (bc' - b'c,\ a'c - ac',\ ab' - a'b)$$

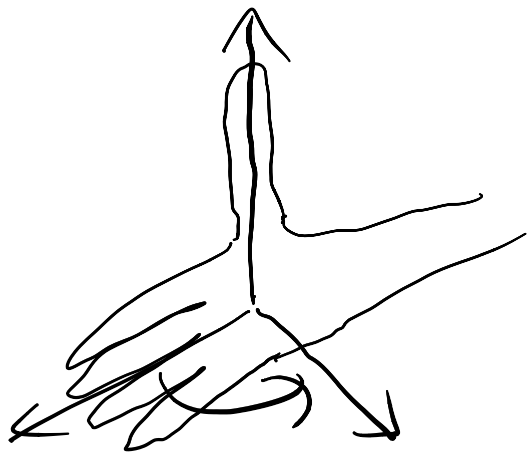
<u>Definition</u>: For any two vectors $\vec{a} = (a, b, c)$ & $\vec{a}' = (a', b', c') \in \mathbb{R}^3$ we define the "cross product vector"

$$\vec{a} \times \vec{a}' := (bc' - b'c,\ a'c - ac',\ ab' - a'b).$$

See HW 2 Problem 4

Meaning: The vector $\vec{a} \times \vec{a}'$ is simultaneously $\perp$ to $\vec{a}$ & $\vec{a}'$:





RIGHT HAND RULE.

Previous Example:

$$(1,1,1) \times (1,2,3)$$

$$= (3-2,\ 1-3,\ 2-1)$$

$$= (1,-2,1)$$

as expected ✓