HW1 due on Thurs before 12:30.
We will discuss solutions in class.

Quiz 1 on Tues Sept 8 at beginning of class.

Remark: Quizzes are much easier than HW assignments.

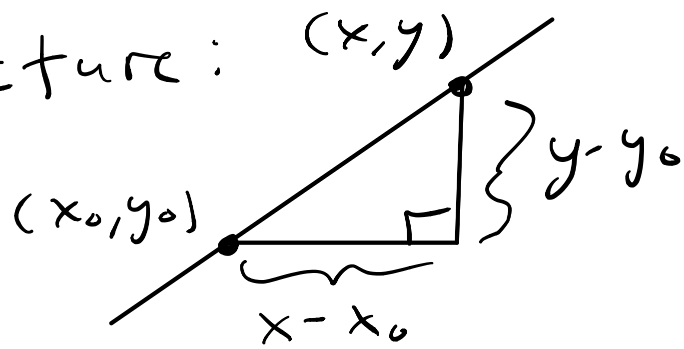
Topic: Lines in the plane $\mathbb{R}^2$.

How to describe a line in the plane?

Old Ways:

- slope & intercept: $y = mx + b$
- point & slope: $(y - y_0) = m(x - x_0)$
  (x_0, y_0) m

Picture:



$m = \text{rise}/\text{run}$
$= (y - y_0)/(x - x_0)$.

- Two points (x_0, y_0) & (x_1, y_1) : First compute slope $m = \dfrac{y_1 - y_0}{x_1 - x_0}$.

Hence $(y - y_0) = m(x - x_0)$

$$(y - y_0) = \frac{y_1 - y_0}{x_1 - x_0}(x - x_0).$$

Issue: vertical line doesn't have a slope! We can fix this multiplying both sides by $x_1 - x_0$:

$$(y - y_0)(x_1 - x_0) = (x - x_0)(y_1 - y_0)$$

This equation also works when line is vertical, i.e., $x_1 = x_0$. Then

$$0 = (x - x_0)\underbrace{(y_1 - y_0)}_{\neq 0}$$

$$0 = x - x_0$$

$$x = x_0.$$

The equation of a vertical line is $x =$ some constant.

To summarize: A line in $\mathbb{R}^2$ is described by an equation of form

$$\boxed{ax + by = c}$$

Jargon: A "linear equation" in two unknowns.

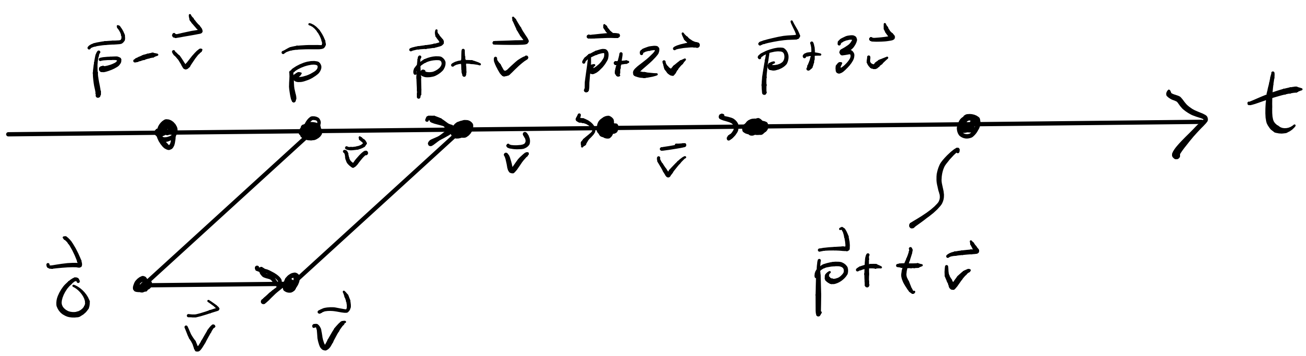
Definition: A linear equation in n unknowns $x_1, x_2, \cdots, x_n$ has the form

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = c$$

for some constants $a_1, a_2, \cdots, a_n, c \in \mathbb{R}$.

New ways to think about lines:

- Point $\vec{p}$ & direction vector $\vec{v}$:

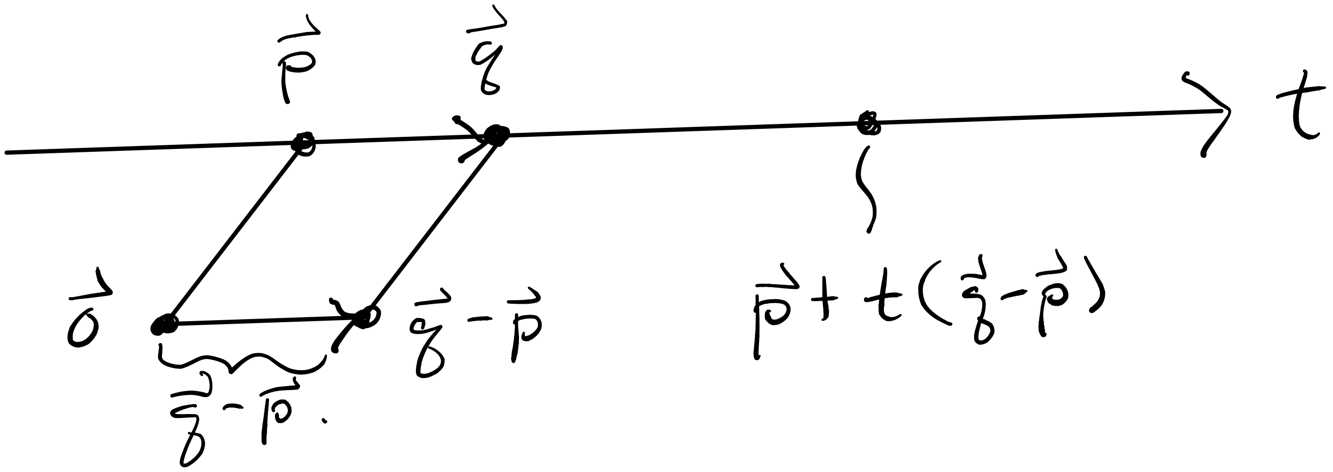


General point on the line has the form $\vec{p} + t\vec{v}$ for some value of t.

Think: $\vec{p}$ = initial position
$\vec{v}$ = velocity
t = time.

The line $= \{\vec{p} + t\vec{v} : t \in \mathbb{R}\}$
= "the set of points of the form $\vec{p}+t\vec{v}$, where t is any real number."

- Two points $\vec{p}$ & $\vec{q}$:



Use $\vec{p}$ as a point
$\vec{q}-\vec{p}$ as a direction vector.

Then we have

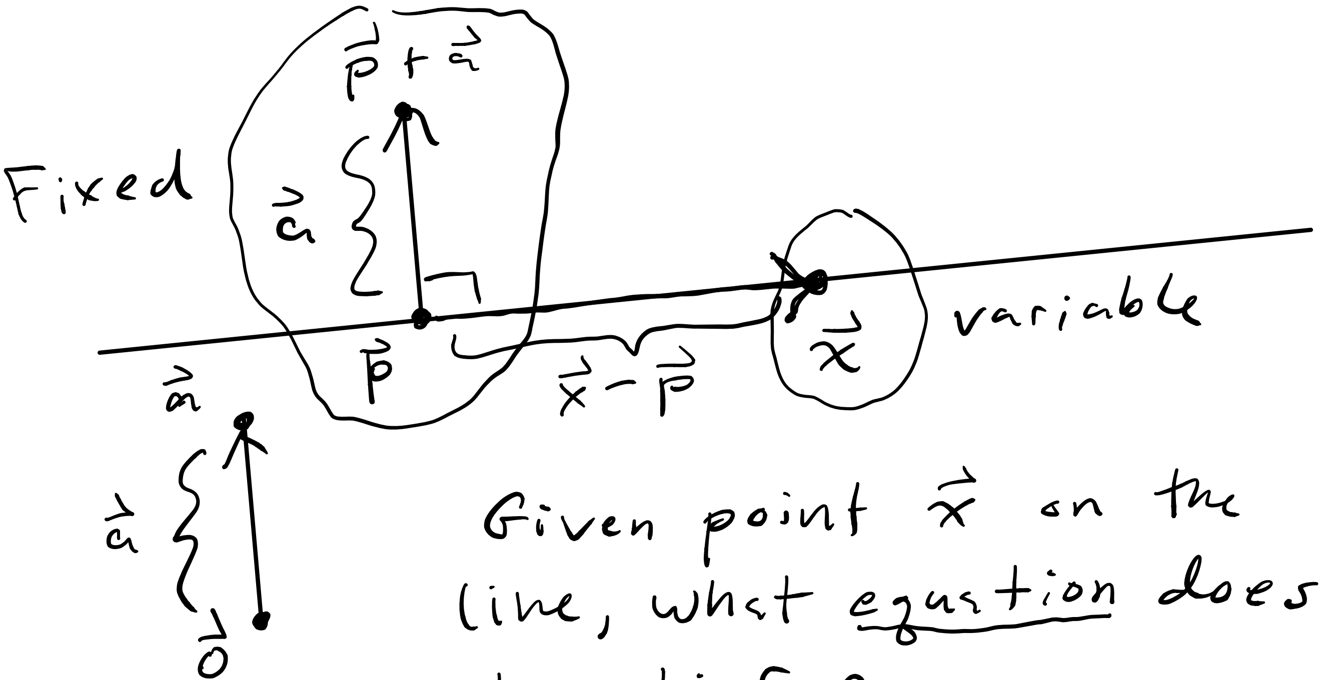
$$\text{line} = \{\vec{p} + t(\vec{q}-\vec{p}) : t \in \mathbb{R}\}$$

$$= \{(1-t)\vec{p} + t\vec{q} : t \in \mathbb{R}\}$$

$$= \{s\vec{p} + t\vec{q} : s,t \in \mathbb{R},\ s+t=1\}$$

ALL THE SAME.

- A point $\vec{p}$ & a "normal vector" $\vec{a}$, i.e., a vector that is perpendicular to the line.



Given point $\vec{x}$ on the line, what <u>equation</u> does it satisfy?

$\vec{x}$ is on the line

$\iff$ vector $\vec{x}-\vec{p}$ is perpendicular to vector $\vec{a}$

$\Longleftrightarrow$ their dot product is zero, i.e.,

$$\vec{a} \cdot (\vec{x} - \vec{p}) = 0$$

This is the equation of the line.

We can also rearrange to get

$$\vec{a} \cdot (\vec{x} - \vec{p}) = 0$$

$$\vec{a} \cdot \vec{x} - \vec{a} \cdot \vec{p} = 0$$

$$\vec{a} \cdot \vec{x} = \vec{a} \cdot \vec{p}$$

fixed vector

variable point.

Fixed number

Compare to the equation

$$ax + by = c$$

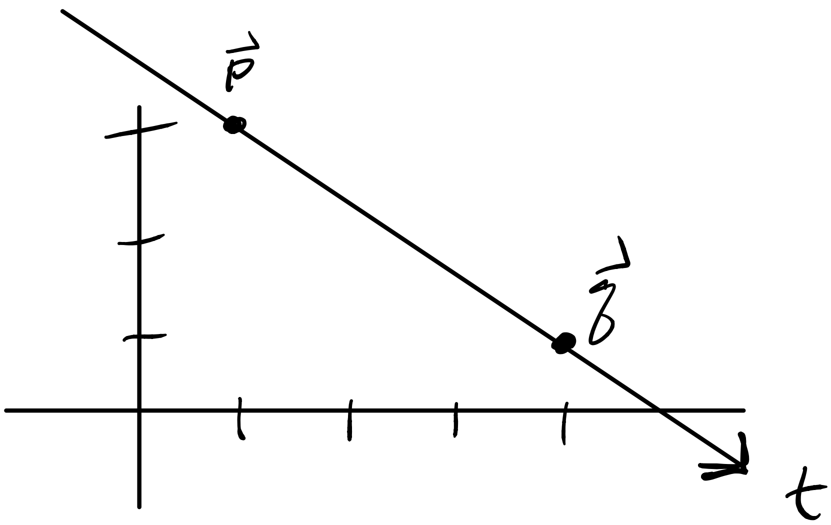
$$\begin{pmatrix} a \\ b \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = c$$

$$\vec{a} \cdot \vec{x} = c. \qquad \text{SAME.}$$

Idea: Any line of this form is $\perp$ to the vector $\vec{a}$.

Examples: Let ℓ be the line containing points $\vec{p} = \begin{pmatrix}1\\3\end{pmatrix}$ & $\vec{q} = \begin{pmatrix}4\\1\end{pmatrix}$.

Parametrized Form:



$$\ell = \{ \vec{p} + t(\vec{q} - \vec{p}) : t \in \mathbb{R} \}$$
$$= \{ (1-t)\vec{p} + t\vec{q} : t \in \mathbb{R} \}$$

Express this equation in the form

$$\vec{a} \circ \vec{x} = c.$$

[Remark: $\vec{p} + t(\vec{q} - \vec{p})$
$= \vec{p} + t\vec{q} - t\vec{p} = (1-t)\vec{p} + t\vec{q}$.]

Ideas ?

- Equation has the form $ax+by=c$ for some <u>unknown</u> constants a, b, c. Plug in the points $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ or $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ to get 2 equations:

$$\begin{cases} a + 3b = c, \\ 4a + b = c. \end{cases}$$

Now "solve" for a, b, c ....
Save this method for later.

- The general point on the line is

$$(1-t)\vec{p} + t\vec{q} = (1-t)\begin{pmatrix} 1 \\ 3 \end{pmatrix} + t\begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1-t \\ 3(1-t) \end{pmatrix} + \begin{pmatrix} 4t \\ t \end{pmatrix}$$

$$= \begin{pmatrix} 1-t+4t \\ 3(1-t)+t \end{pmatrix}$$

$$= \begin{pmatrix} 1-t+4t \\ 3-3t+t \end{pmatrix} = \begin{pmatrix} 1+3t \\ 3-2t \end{pmatrix}$$

In x, y - coordinates we have

$$\begin{cases} x = 1 + 3t, \\ y = 3 - 2t. \end{cases}$$

"Eliminate" t:

$x = 1 - 3t$
$x - 1 = 3t$
$t = (x-1)/3$

$y = 3 - 2t$
$y - 3 = 2t$
$t = (y-3)/(-2)$

$$(x-1)/3 = (y-3)/(-2)$$

$$(x-1)(-2) = (y-3)(3)$$

$$-2x + 2 = 3y - 9$$

$$-2x - 3y = -9 - 2$$

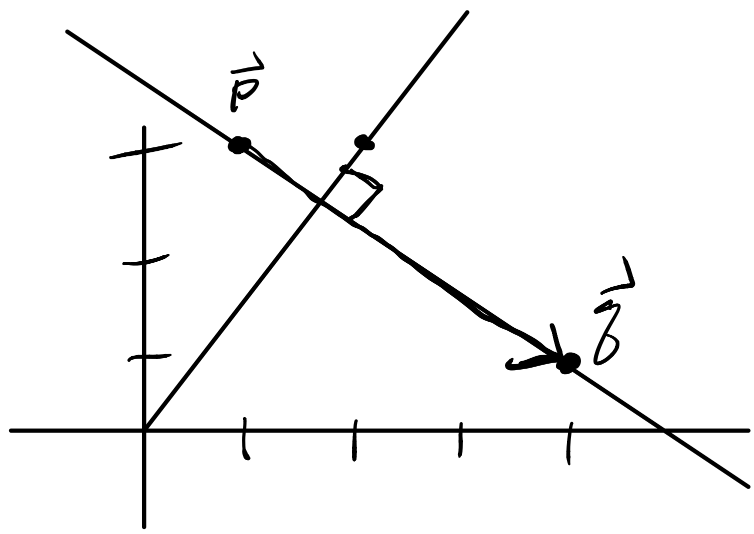
$$2x + 3y = 11$$

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = 11 \quad \checkmark$$

That works, but there's a better way:

- Better Way:

To find a vector $\perp$ to the line



Direction vector

$$\vec{q} - \vec{p} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}.$$

Can you think of any vector that is perpendicular to $(3, -2)$?

[Trick ("Negative Reciprocal"):
vector (a, b) is $\perp$ to $\pm(b, -a)$.]

Application: $(2, 3) \perp (3, -2)$

$(-2, -3) \perp (3, -2)$.

I'll pick $\vec{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

So the line must have the form

$$\vec{a} \circ \vec{x} = c$$

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} \circ \begin{pmatrix} x \\ y \end{pmatrix} = c$$

$$2x + 3y = c$$

for some constant c.

Plug in any point to get c.

e.g. $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ gives

$$\begin{aligned} c &= 2x + 3y \\ &= 2(1) + 3(3) = 11 \end{aligned}$$

This method was quicker!

Hints for Problem 5(b).

For any nonzero vector $\vec{a} \in \mathbb{R}^2$ and for any constant $c \in \mathbb{R}$, I claim that the line

$$\vec{a} \cdot \vec{x} = c$$

is $\perp$ to the line $t\vec{a}$.

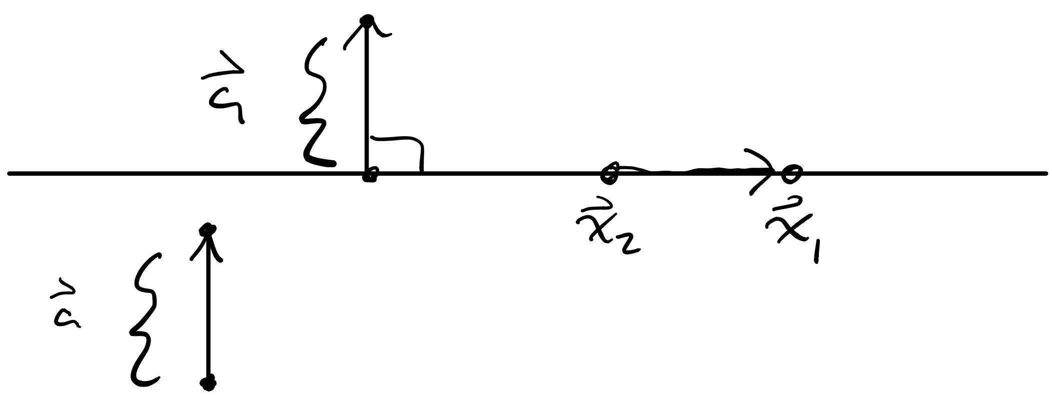
Idea: For any two points $\vec{x}_1$ & $\vec{x}_2$ on the line we want to show that

$$\vec{x}_1 - \vec{x}_2 \perp \vec{a}$$

i.e., that the dot product is zero:

$$\vec{a} \cdot (\vec{x}_1 - \vec{x}_2) = 0.$$

Picture:



Summary: Assuming that $\vec{x}_1$ & $\vec{x}_2$ satisfy equation $\vec{a} \cdot \vec{x} = c$, show that $\vec{a} \cdot (\vec{x}_1 - \vec{x}_2) = 0$.