HW 4 due Thursday before class.

Office Hours Today: 1:45 – 2:15

Tomorrow: 4 – 5.

Recall, if A is $l \times m$ & B is $m \times n$ then the matrix AB exists. It has shape $l \times n$ and is defined by requiring that

$$(AB)\vec{x} = A(B\vec{x})$$

for all $\vec{x} \in \mathbb{R}^n$.

Meaning: AB is the matrix of the linear function $A \circ B$.

How to Compute? Memorize!

(ith row AB) = (ith row A) B

(jth col AB) = A(jth col B)

(ij entry AB) = (ith row A)(jth col B)

dot product!

One more formula:

$$AB = \sum_{k=1}^{m} (k\text{th } \underline{\text{col}}\ A)(k\text{th row } B)$$

NOT dot product!

Remark: If $\vec{x}, \vec{y} \in \mathbb{R}^n$ are $n \times 1$ column vectors, then

$\vec{x}^T \vec{y}$ is 1×1 matrix, i.e., scalar.

It is just the dot product.

If $\vec{x} \in \mathbb{R}^m$ is $m \times 1$

$\vec{y} \in \mathbb{R}^n$ is $n \times 1$

then $\vec{x}\vec{y}^T$ is an $n \times m$ matrix, it is <u>not</u> a scalar. Matrices of the form (column)(row) are called "rank 1 matrices" because they have rank 1 (one pivot in RREF).

We will see more of them when we discuss "projection."

Example:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 2 & 3 \end{pmatrix}.$$

$$(\text{2nd row } AB)$$

$$= (\text{2nd row } A)\, B$$

$$= \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1-2+0 & 0+2+6 & 1+0+9 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 8 & 10 \end{pmatrix}$$

$$(\text{3rd col } AB)$$

$$= A\, (\text{3rd col } B)$$

$$= \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1+0+3 \\ 1+0+9 \end{pmatrix}$$

$$= \begin{pmatrix} 4 \\ 10 \end{pmatrix}$$

Finally:

$$AB = \sum_{k=1}^{3} (k\text{th } \underline{\text{col}}\ A)(k\text{th } \underline{\text{row}}\ B)$$

$$= \begin{pmatrix} 1 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}\begin{pmatrix} -1 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \end{pmatrix}\begin{pmatrix} 0 & 2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 1 & 0 \\ -2 & 2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 2 & 3 \\ 0 & 6 & 9 \end{pmatrix}$$

$$= \begin{pmatrix} 1-1+0 & 0+1+2 & 1+0+3 \\ 1-2+0 & 0+2+6 & 1+0+9 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 3 & 4 \\ -1 & 8 & 10 \end{pmatrix} \checkmark$$

Rules of Matrix Arithmetic:

Let A, B, C be matrices,

Let s, t be scalars,

Then the following formulas hold (whenever they are defined):

$$A + B = B + A$$
$$(A+B)+C = A+(B+C)$$
$$AO = O \quad \& \quad OA = O$$
$$AI = A \quad \& \quad IA = A$$

[O & I are zero & identity matrices]

$$(s+t)A = sA + tA$$
$$t(A+B) = tA + tB$$
$$A(BC) = (AB)C$$
$$A(B+C) = AB + AC$$
$$(A+B)C = AC + BC$$

Note that this includes vector arithmetic as a special case because:

- vectors are matrices,
- dot product is matrix multiplication.

Warning: $AB \neq BA$ in general.

One more operation: Transpose.

(ij entry A^T) = (ji entry A).

Then I claim:

$$(A+B)^T = A^T + B^T$$

$$(AB)^T = B^T A^T \quad \text{(Surprise!)}$$

Let's check:

Suppose A is $l \times m$, B is $m \times n$, so that AB is $l \times n$.

Then A^T is $m \times l$, B^T is $n \times m$, so that $A^T B^T$ is <u>not defined</u> (unless $l = n$). However, $B^T A^T$ is <u>always</u> defined and has shape

$$\underset{n \times m}{B^T} \cdot \underset{m \times l}{A^T} = \underset{n \times l}{B^T A^T}$$

match ✓

So, $(AB)^T$ & $B^T A^T$ exist

and have the same shape. So they must be equal, right?

Check:

ij entry $(AB)^T$

= ji entry AB

= (jth row A)(ith col B)

dot product is symmetric

= (ith row B^T)(jth col A^T)

= ij entry $B^T A^T$ ✓

Example:

$$A = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix}$$

$$AB = \begin{pmatrix} 1+0+3 & 0+2+6 \end{pmatrix} = \begin{pmatrix} 4 & 8 \end{pmatrix}$$

$$(AB)^T = \begin{pmatrix} 4 \\ 8 \end{pmatrix}$$

On the other hand:

$$A^T = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad B^T = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

$$B^T A^T = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1+0+3 \\ 0+2+6 \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \end{pmatrix} \checkmark$$

Invertibility:

We say that A & B are inverse matrices if

$$AB = I \quad \& \quad BA = I.$$

Remarks:

- If B exists, it is <u>unique</u> so we can call it <u>the</u> inverse of A & give it a special notation:

"A^{-1}" = <u>the</u> inverse of A

- If A^{-1} exists then A must be SQUARE. (Subtle.)
- IF A, B are SQUARE and if $AB = I$ then necessarily $BA = I$. (Subtle and hard to prove!)

Thus we only have to check <u>one</u> of the equations

$$AB = I$$
$$BA = I$$

How to compute the inverse (or show that is doesn't exist):

<u>Some</u>times we can use geometry.

Examples:

- Let R_θ be 2×2 matrix that <u>rotates</u> c.c.w. by angle θ.

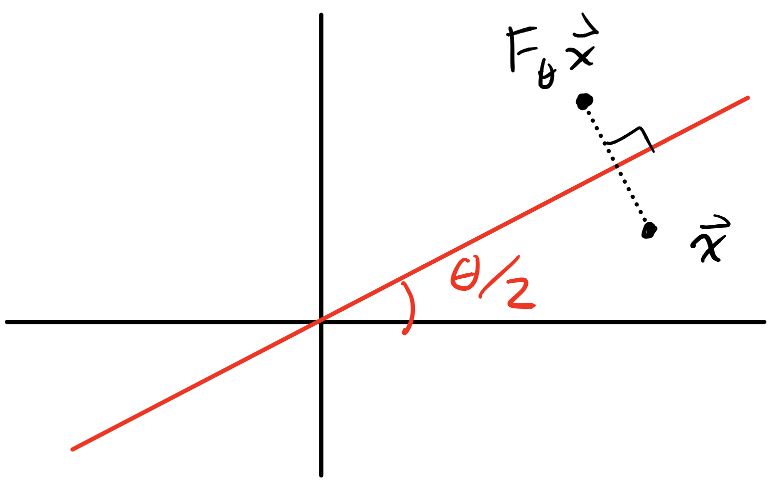
  Then R_θ^{-1} exist and

$$R_\theta^{-1} = \text{rotate clockwise by } \theta$$
$$= \text{rotate c.c.w. by } -\theta$$
$$= R_{-\theta}.$$

Explicitly:

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

- Let F_θ be 2×2 matrix that <u>reflects</u> across the line with angle $\theta/2$



Then F_θ^{-1} = do the same reflection again

$= F_\theta$

Remark: Multiply both sides by F_θ to get

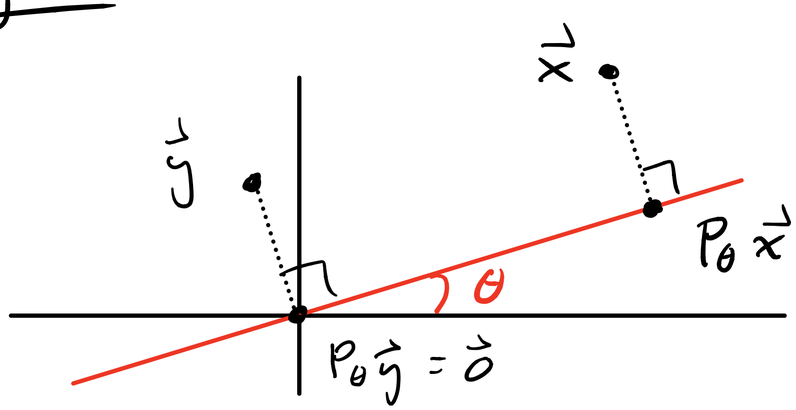
$$F_\theta = F_\theta^{-1}$$

$$F_\theta F_\theta = F_\theta F_\theta^{-1}$$

$$(F_\theta)^2 = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Meaning: Performing the same reflection twice is the same as "doing nothing."

- Let P_θ be 2×2 matrix that projects onto the line of slope θ:



I claim that P_θ^{-1} does not exist.

Why not?

Let $\vec{y}$ be any nonzero vector that projects to zero:

$$P_\theta \vec{y} = \vec{0}$$

But then the inverse P_θ^{-1} cannot exist. If it did, then we could multiply on the left to get

$$P_\theta^{-1} P_\theta \vec{y} = P_\theta^{-1} \vec{0}$$

$$I \vec{y} = \vec{0}$$

$$\vec{y} = \vec{0}$$

Contradiction!

[See HW4.5(b).]

Other times we have to use algebra.

Claim: Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with

determinant $\det(A) = ad - bc$.

Then A^{-1} exists $\iff \det(A) \neq 0$,

in which case, we have

$$\boxed{A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.}$$

Proof: We only need to check

that $A^{-1}A = I$:

$$A^{-1}A = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$= \frac{1}{\det(A)} \begin{pmatrix} ad - bc & bd - bd \\ -ac + ac & -bc + ad \end{pmatrix}$$

$$= \frac{1}{\cancel{\det(A)}} \begin{pmatrix} \cancel{ad - bc} & 0 \\ 0 & \cancel{ad - bc} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \checkmark$$

For general $n \times n$ matrix A, it is still true that

$$A^{-1} \text{ exists} \iff \det(A) \neq 0,$$

but the explicit formula is too terrible to memorize, so instead we use Gaussian elimination to compute it.

The Algorithm:

- Form the "augmented" matrix

$$(A \mid \underbrace{I}_{n \times n \text{ identity}})$$

- Put this matrix in RREF.
- If you obtain a matrix of the form

$$\left(I \mid B \right)$$

I: $n \times n$ identity

then $B = A^{-1}$ is the (unique) inverse.

- If you don't obtain a matrix of this form then the inverse A^{-1} <u>does not exist</u>.

Example: Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 2 & 4 \end{pmatrix}$.

To compute A^{-1} we form the "augmented" matrix $(A \mid I)$ and then compute the RREF:

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 3 & 4 & 0 & 1 & 0 \\ 1 & 2 & 4 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right) \quad \begin{array}{l} ② = ② - 2① \\ ③ = ③ - 1① \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right) \quad ② = -1②$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 4 & 0 & -3 \\ 0 & 1 & 0 & 4 & -1 & -2 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right) \quad \begin{array}{l} ① = ① - 3③ \\ ② = ② - 2③ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -4 & 2 & 1 \\ 0 & 1 & 0 & 4 & -1 & -2 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right) \quad ① = ① - 2②$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -4 & 2 & 1 \\ 0 & 1 & 0 & 4 & -1 & -2 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right) \quad \text{DONE.}$$

We conclude that the matrix A is invertible, with inverse

$$A^{-1} = \begin{pmatrix} -4 & 2 & 1 \\ 4 & -1 & -2 \\ -1 & 0 & 1 \end{pmatrix}.$$

Check:

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 2 & 4 \end{pmatrix} \begin{pmatrix} -4 & 2 & 1 \\ 4 & -1 & -2 \\ -1 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -4+8-3 & 2-2+0 & 1-4+3 \\ -8+12-4 & 4-3+0 & 2-6+4 \\ -4+8-4 & 2-2+0 & 1-4+4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \checkmark$$

Next time I'll tell you why this method works.