Quiz 3 solutions are up.
Due to medical issues, HW 4 is delayed until Thurs Oct 22.
This means all HW and Quizzes are delayed 1 week.

Recall: A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is called "linear" if

- it preserves vector addition:

$$f(\vec{u} + \vec{v}) = f(\vec{u}) + f(\vec{v})$$

- it preserves scalar multiplication:

$$f(t\vec{v}) = t f(\vec{v}).$$

[In particular, taking $t = 0$ gives

$$f(\vec{0}) = \vec{0}.$$]

We can also summarize this definition in one step:

$$f(s\vec{u} + t\vec{v}) = s f(\vec{u}) + t f(\vec{v})$$

"f preserves linear combinations"

Geometric Meaning: A linear function sends $\vec{0}$ to $\vec{0}$ and sends parallelograms to parallelograms.

Theorem: If $f : \mathbb{R}^n \to \mathbb{R}^m$ is linear, then there exists an $m \times n$ matrix $[f]$ such that

$$f(\vec{x}) = [f]\vec{x} \quad \text{for all } \vec{x} \in \mathbb{R}^n$$

matrix times column vector.

Proof: Let $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n \in \mathbb{R}^n$ be the "standard basis vectors":

$$\vec{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0).$$

ith position.

So for any vector $\vec{x} = (x_1, x_2, \ldots, x_n)$ we observe that

$$\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_n\vec{e}_n$$

$= x_1(1, 0, \ldots, 0)$

$\quad + x_2(0, 1, 0, \ldots, 0)$

$\quad\quad + \cdots + x_n(0, \ldots, 0, 1)$ ✓

Let $[f]$ be the $m \times n$ matrix whose jth column vector is $f(\vec{e}_j) \in \mathbb{R}^m$:

$$[f] = \underbrace{\left(f(\vec{e}_1) \cdots f(\vec{e}_n) \right)}_{n} \Big\} m$$

Then by the definition of linearity & matrix times column vector, we have

$$[f]\vec{x} = \left(f(\vec{e}_1) \cdots f(\vec{e}_n) \right) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

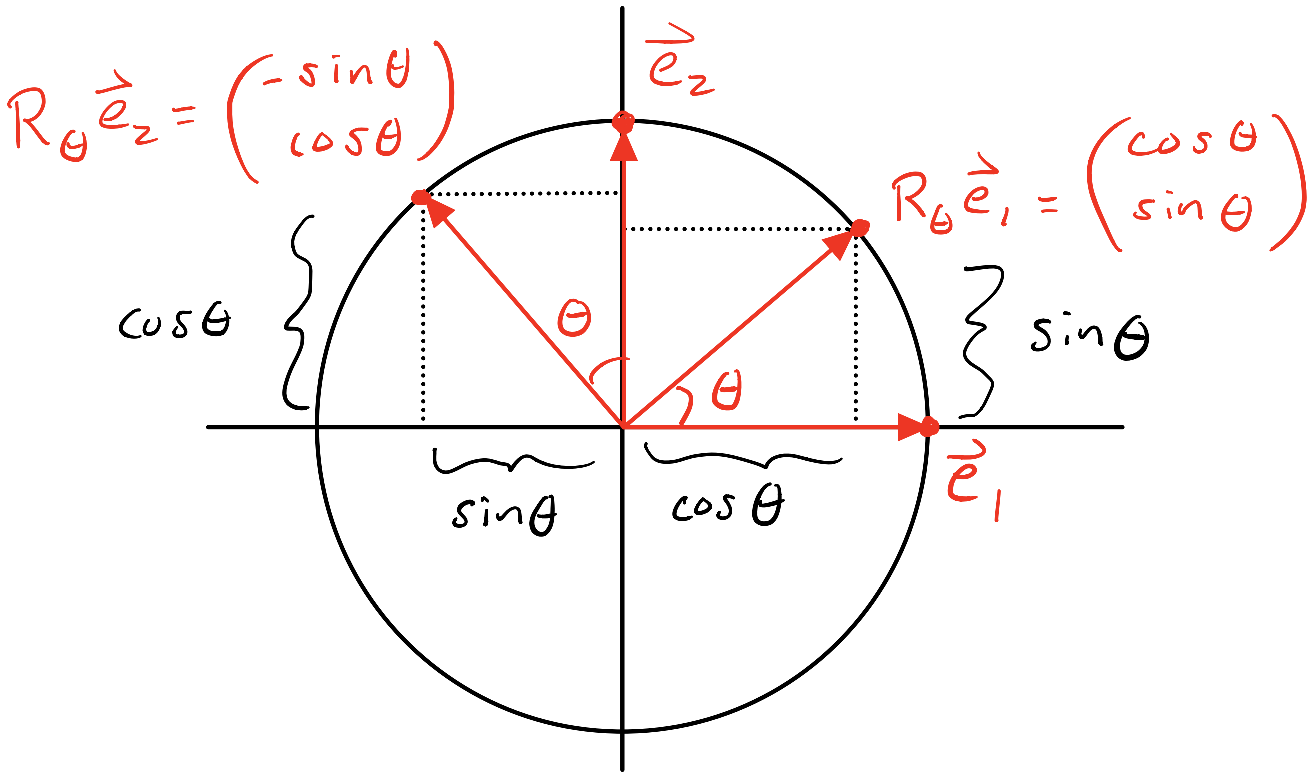
$$= x_1 f(\vec{e}_1) + \cdots + x_n f(\vec{e}_n) \qquad \text{def}$$

$$= f(x_1\vec{e}_1 + \cdots + x_n\vec{e}_n) \qquad \text{def}$$

$= f(\vec{x})$ as desired. ✓

Example: Let $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be the function that rotates by angle θ counter-clockwise around the origin, which is a linear function.

Let's find the matrix:



It follows that the matrix is

$$R_\theta = \left(R_\theta \vec{e}_1 \; \middle| \; R_\theta \vec{e}_2 \right)$$

$$= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

To rotate a general point $\vec{x} = (x, y)$ we multiply by the matrix:

$$R_\theta \vec{x} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix}$$

That's pretty good!

Example from last time:

R_{90° = rotate c.c.w. by 90°

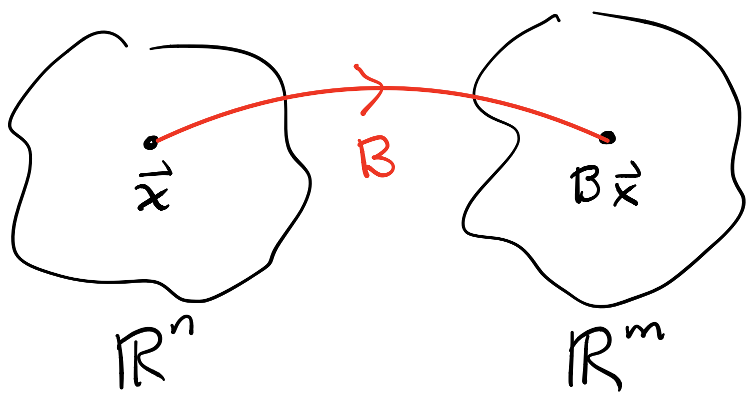
$$= \begin{pmatrix} \cos 90^\circ & -\sin 90^\circ \\ \sin 90^\circ & \cos 90^\circ \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \checkmark$$

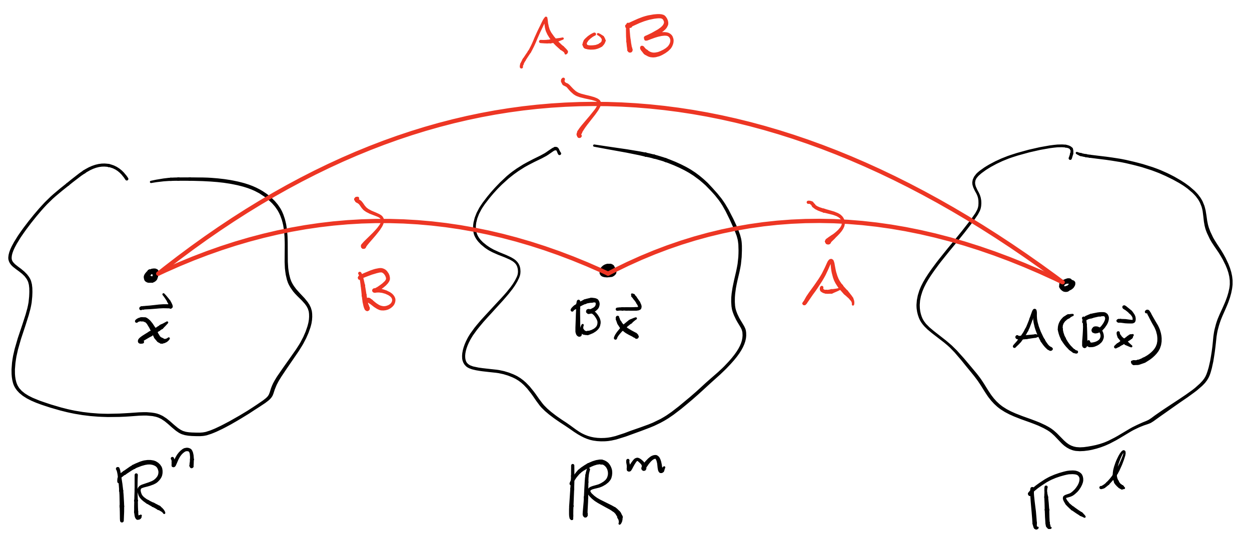
So any linear function is a matrix.

Conversely, any $m \times n$ matrix B

can be thought of as a function from $\mathbb{R}^n \to \mathbb{R}^m$:



And you will show that this function is linear. If $A: \mathbb{R}^m \to \mathbb{R}^l$ is any $l \times m$ matrix, then we consider the composite function $A \circ B: \mathbb{R}^n \to \mathbb{R}^l$:



The function $A \circ B$ is also linear, so it corresponds to an $l \times n$ matrix:

$$[A \circ B] = \underbrace{\begin{pmatrix} A(B\vec{e}_1) & \cdots & A(B\vec{e}_n) \end{pmatrix}}_{n} \Big\} \ell$$

Important Definition:

If A is $\ell \times m$ matrix & B is $m \times n$ then we define $\ell \times n$ matrix AB as the matrix of the composite function:

$$AB := [A \circ B]$$

= the matrix of the linear function $A \circ B$.

In other words, we have

$$A(B\vec{x}) = (AB)\vec{x}$$

for all vectors $\vec{x} \in \mathbb{R}^n$.

Fine. But how do we compute this matrix?

Well, we know that

$$\begin{aligned} j\text{th col } AB &= (AB)\vec{e}_j \\ &= A(B\vec{e}_j) \\ &= A(j\text{th col } B), \end{aligned}$$

which is already defined. :)

It follows that

$$\begin{aligned} &i,j \text{ entry of } AB \\ &\quad = i\text{th entry of } A(j\text{th col } B) \\ &\quad = (i\text{th row } A)(j\text{th col } B) \end{aligned}$$

dot product.

Examples:

- Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 3 & 1 \\ 1 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 2 & -1 \\ 0 & 1 \\ 1 & 2 \end{pmatrix}$

Compute AB:

$$AB = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 3 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 0 & 1 \\ 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2+0+0 & -1+2+0 \\ 4+0+1 & -2+3+2 \\ 2+0+1 & -1+0+2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 \\ 5 & 3 \\ 3 & 1 \end{pmatrix}$$

But note that BA is <u>not defined</u>.

Reason: # cols $B \neq$ # rows A.

In other words, the function

$B \circ A$ is not defined because

$$A : \mathbb{R}^3 \to \mathbb{R}^3$$
$$B : \mathbb{R}^2 \to \mathbb{R}^3$$

Exmphasis: If A is $k \times l$
B is $m \times n$

then AB is only defined when

$\ell = m$

$\#$ cols A = $\#$ rows B.

In which case, AB is $k \times n$.

$$\underset{k \times \ell}{A} \cdot \underset{m \times n}{B} = \underset{k \times n}{AB}$$

these must match

- Let $R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

  = rotate $90°$ c.c.w.

  $F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

  = reflect across line $x = y$.

Compute RF & FR:

$$RF = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0-1 & 0+0 \\ 0+0 & 1+0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$FR = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0+1 & 0+0 \\ 0+0 & -1+0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Observe $RF \neq FR$.

Geometrically:

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix}$$

"reflect across line $x = 0$"

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$$

"reflect across line $y = 0$"

We conclude that

"reflect across $x = y$ and <u>then</u> rotate ccw $90°$"

$= RF = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} =$ "reflect across $x = 0$"

"rotate ccw $90°$ and <u>then</u> reflect across line $x=y$"

$= FR = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} =$ "reflect across $y=0$"

These geometric facts are not so obvious, but the algebra was easy to compute. Slogan:

"Algebra is smarter than geometry"

Next Topic: Inverse Matrices.

Let $f: \mathbb{R}^n \to \mathbb{R}^m$

$g: \mathbb{R}^m \to \mathbb{R}^n$ be functions.

We say f & g are inverses if

$f \circ g: \mathbb{R}^m \to \mathbb{R}^m$

$g \circ f: \mathbb{R}^n \to \mathbb{R}^n$

are both the identity (do-nothing) function. Now let's suppose that f & g are <u>linear</u> functions with corresponding matrices:

$[f]$ $m \times n$

$[g]$ $n \times m$

If f & g are inverse functions, then this implies:

$[f][g] = I$ ($m \times m$ identity)

$[g][f] = I$ ($n \times n$ identity)

[Subtle Fact: In this case we <u>must</u> have $m = n$, so $[f]$ & $[g]$ are <u>square</u> matrices of the same size.]

Definition: Let A be a square $n \times n$ matrix.

We say that A is <u>invertible</u> if there exists a square $n \times n$ matrix B such that

$$AB = I \quad (n \times n)$$
$$BA = I \quad (n \times n).$$

In this case, the matrix B is unique, and we call it

"A^{-1}" = <u>the</u> inverse matrix of A

Examples:

- Let $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

What is the inverse matrix? Geometrically, we see that

R_θ^{-1} = rotate <u>clockwise</u> by θ

= rotate c.c.w. by $-\theta$

$$= R_{-\theta}$$

$$= \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Check :

$$R_\theta R_{-\theta} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \cos\theta\sin\theta \\ \cos\theta\sin\theta - \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \quad \checkmark$$

The algebra & the geometry agree, as they should. :)

- Let F = reflect across line $x = y$
  What is the inverse ?

Claim: $F^{-1} = F$.

Indeed, to undo the reflection we should just do the same reflection again!

Check that the algebra works:

$$FF = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0+1 & 0+0 \\ 0+0 & 1+0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \checkmark$$

[Remark: The identity matrix is not the only matrix that is equal to its own inverse; any reflection also has this property.]

- The $n \times n$ zero matrix O is never invertible because for any $n \times n$ matrix A we have

$$OA = O \neq I \quad \& \quad AO = O \neq I.$$

- I claim that the matrix

$$N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

is <u>not</u> invertible. To see this suppose that we have

$$NA = I$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

for some matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

But then we compute

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix},$$

which is a contradiction because the bottom-right entry is never equal to 1.

[Remark : There exist nonzero matrices that are not invertible. In fact, we will see later that any "projection matrix" is not invertible.

The theory of matrix arithmetic is quite a bit more interesting than the usual arithmetic of numbers (i.e. 1×1 matrices).]