HW 3 is due now.

Today: HW3 Discussion
+ Review for Quiz 3.

General Remarks:

Consider a system of $m$ linear equations in $n$ unknowns:

$$\begin{cases} \vec{a}_1 \cdot \vec{x} = b_1, \\ \vec{a}_2 \cdot \vec{x} = b_2, \\ \quad \vdots \\ \vec{a}_m \cdot \vec{x} = b_m, \end{cases}$$

where $\vec{x} = (x_1, x_2, \dots, x_n)$ are the variables & $\vec{a}_i = (a_{i1}, a_{i2}, \dots, a_{im})$ are some constant vectors.

By Problem 4, the set of solutions is "flat." What does this mean?

The solutions live in n-dim space:

$$S \subseteq \mathbb{R}^n$$

$$S = \{ \vec{x} \in \mathbb{R}^n : \vec{a}_i \circ \vec{x} = b_i \text{ for all } i \}$$

What kind of shape is $S$?

Claim: If $\vec{p}$ & $\vec{q}$ are two points in $S$ then the whole line

$$(1-t)\vec{p} + t\vec{q}$$

are in the set

is in $S$.

Proof: Assume $\vec{p}, \vec{q} \in S$. By definition this means that

$$\vec{a}_i \circ \vec{p} = b_i \quad \& \quad \vec{a}_i \circ \vec{q} = b_i$$

for all $i = 1, 2, \ldots, m$. Then for any scalar $t$ and for any $i = 1, 2, \ldots, m$ we have

$$\vec{a}_i \circ \left[ (1-t)\vec{p} + t\vec{q} \right]$$

$$= (1-t)\,\vec{a_i}\cdot\vec{p} + t\,\vec{a_i}\cdot\vec{q}$$

$$= (1-t)\,b_i + t\,b_i$$

$$= b_i - \cancel{t b_i + t b_i}$$

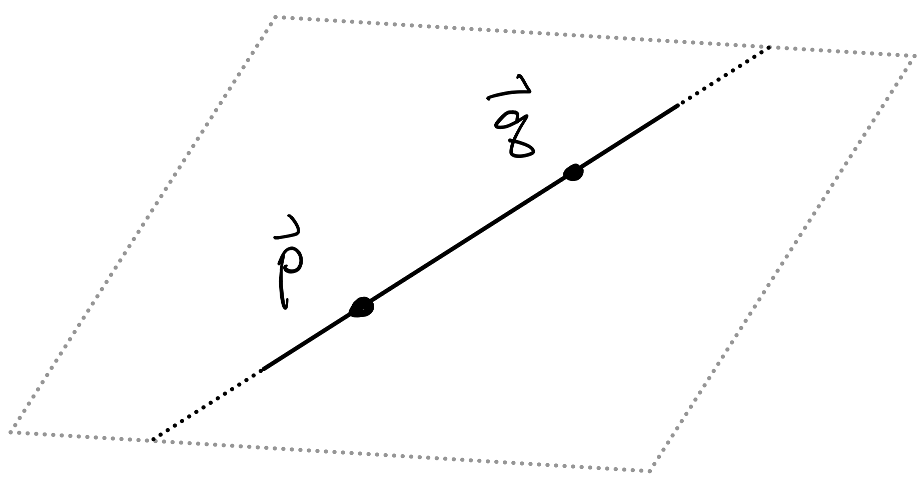
$$= b_i$$

By definition, this means that

$$(1-t)\vec{p} + t\vec{q} \in S.$$

"The point $(1-t)\vec{p} + t\vec{q}$ is in the set of solutions of the system."

Picture: The solution set is "flat."



Conclusion: The solution set of a system of $m$ linear equations in $n$ unknowns is either empty, or it is a "$d$-plane" for some $d$.

Meaning:

$$S = \{\vec{p} + t_1\vec{u}_1 + \cdots + t\vec{u}_d : \text{scalars } t\}$$

where $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_d \in \mathbb{R}^n$ are independent vectors living in $n$-dimensional space.

How to compute $d$ & find some vectors $\vec{p}, \vec{u}_1, \vec{u}_2, \ldots, \vec{u}_d$?

Answer: RREF.

Perform Gaussian elimination on the system. If we get some equation of the form

$$0x_1 + 0x_2 + \cdots + 0x_n = c$$

where $c \neq 0$, then there is no solution. Otherwise, the solution is a $d$-plane where

$d$ = # free variables in the RREF.

Example: Problem 2.

$$\begin{cases} x_1 + 2x_2 + x_3 + 0 + 2x_5 = 1, \\ x_1 + 2x_2 + 2x_3 - 3x_4 + 3x_5 = 1, \\ x_1 + 2x_2 + 0 + 3x_4 + 2x_5 = 3. \end{cases}$$

$$\Big\downarrow \text{RREF}$$

$$\begin{cases} x_1 + 2x_2 + 0 + 3x_4 + 0 = -1, \\ 0 + 0 + x_3 - 3x_4 + 0 = -2, \\ 0 + 0 + 0 + 0 + x_5 = 2. \end{cases}$$

[See HW3 solutions for the steps.]

Pivot variables : $x_1, x_3, x_5$

Free variables : $x_2, x_4$

Rename : $s = x_2$ & $t = x_4$

The solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -1-2s-3t \\ s \\ -2+3t \\ t \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ 0 \\ -2 \\ 0 \\ 2 \end{pmatrix} + s \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -3 \\ 0 \\ 3 \\ 1 \\ 0 \end{pmatrix}$$

$$= \vec{p} + s\vec{u} + t\vec{v}$$

A 2-plane living in 5-dim space.

In this case

$m = \#$ eqns $= 3$

$n = \#$ variables $= 5$

$d = \#$ free variables $= 2$.

In general, let

$r = \#$ pivot variables

$d = \#$ free variables

so that

$$r + d = \text{total } \# \text{ variables} = n$$

There can be at most one pivot in each row, so that

$$0 \leq r \leq m$$

$$0 \leq \# \text{pivot vars} \leq \# \text{equations}$$

Apply the formula $r + d = n$

$$-m \leq -r \leq 0$$

$$n - m \leq n - r \leq n$$

$$n - m \leq d \leq n.$$

Assuming the solution is not empty,

the dimension of the solution must satisfy

$$n - m \leq d \leq n$$

If the system was chosen "randomly" I have claimed that $n - m = d$, i.e.,

$$\text{dim of solutions} = \#\text{variables} - \#\text{equations}.$$

More precisely, this happens when

$$r = m,$$

i.e., when there is a pivot in <u>every row</u>, i.e., when the row vectors of the system have <u>no linear relations</u>.

"no row relations"

Example: Problem 2

The row vectors of matrix

$$\begin{pmatrix} 1 & 2 & 1 & 0 & 2 & 1 \\ 1 & 2 & 2 & -3 & 3 & 1 \\ 1 & 2 & 0 & 3 & 2 & 3 \end{pmatrix}$$

are independent, i.e., they point in "3 different directions." This is why we got a pivot in every row of RREF.

Example: Problem 1.

$$\begin{cases} x + 2y + 3z = 4, \\ x + 2y + 4z = 6, \\ x + 2y + 5z = 8. \end{cases}$$

$$\Big\downarrow \text{RREF}$$

$$\begin{cases} x + 2y + 0 = -2, \\ 0 + 0 + z = 2, \\ 0 + 0 + 0 = 0. \end{cases}$$

Pivot variables : $x, z$

Free variables : $y$.

Rename : $t = y$.

The solution is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2-2t \\ t \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix} + t \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

$$= \vec{p} + t\vec{u}$$

A 1-plane (i.e., a line) living in 3-dimensional space $\mathbb{R}^3$.

$m$ = # equations = 3

$n$ = # variables = 3

$r$ = # pivot variables = 2

$d$ = # free variables = 1.

In this case we have

$$n - m < d$$

because

$$r < m.$$

$$\# \text{ pivots} < \# \text{ rows}.$$

We did not get a pivot in every row. Why not?

There must be a "row relation" in the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 6 \\ 1 & 2 & 5 & 8 \end{pmatrix}.$$

How can we find this relation?

$$r\,(\text{row } 1) + s\,(\text{row } 2) + t\,(\text{row } 3)$$
$$= (0 \ \ 0 \ \ 0 \ \ 0)$$

for some $r, s, t$ not all zero.

Find $r, s, t$.

TRICK : Problem 3.

Flip the matrix on its side:

$$A^T = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \\ 4 & 6 & 8 \end{pmatrix}.$$

"the transpose of A"

Compute the RREF of $A^T$:

$$\text{RREF}(A^T) = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

So what? In $\text{RREF}(A^T)$ we observe that

$$(\text{col } 3) = -1(\text{col } 1) + 2(\text{col } 2)$$

$$\begin{pmatrix}-1\\2\\0\\0\end{pmatrix} = -1\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + 2\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

EASY.

TRICK: Column relations are unchanged by row operations.
So the <u>same</u> column relation must hold in $A^T$:

$$\begin{pmatrix}1\\2\\5\\8\end{pmatrix} = -1\begin{pmatrix}1\\2\\3\\4\end{pmatrix} + 2\begin{pmatrix}1\\2\\4\\6\end{pmatrix}$$

[<u>Maybe</u> you could have found this relation by trial-and-error, but RREF is a systematic method.]

Finally:

column relations in $A^T$ = row relations in $A$.

Conclusion: In the matrix

$$A = \begin{array}{c} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 6 \\ 1 & 2 & 5 & 8 \end{pmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array}$$

we have row relation

$$3 = -1\,(1) + 2\,(2)$$

$$1\,(1) - 2\,(2) - 1\,(3) = (0\ 0\ 0\ 0)$$

$$(r, s, t) = (1, -2, -1).$$

And <u>this</u> is why RREF(A) has a row of zeroes

In other words, the system

$$\begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{cases} x + 2y + 3z = 4, \\ x + 2y + 4z = 6, \\ x + 2y + 5z = 8, \end{cases}$$

really only has 2 independent equations.

Problem 5:

Let $U \leq \mathbb{R}^n$ be a "d-dimensional subspace of $\mathbb{R}^n$."

Jargon:

"d-dim subspace" = "d-plane passing through $\vec{0}$"

Why the fancy name? Never mind.

So, $U \leq \mathbb{R}^n$ d-dim subspace means

$$U = \{ t_1 \vec{u}_1 + \cdots + t_d \vec{u}_d : t_1, \ldots, t_d \in \mathbb{R} \}$$

for some independent vectors

$$\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_d \in \mathbb{R}^n$$

[This implies that $d \leq n$.]

Now let $U^\perp \leq \mathbb{R}^n$ be the "orthogonal complement" of $U$.

Meaning:

For all $\vec{x} \in U$ & $\vec{y} \in U^\perp$ we have

$$\vec{x} \cdot \vec{y} = 0.$$

Then I claim that

$$\dim U + \dim U^\perp = \dim \mathbb{R}^n$$

$$d + \dim U^\perp = n$$

$$\dim U^\perp = n - d.$$

Example: Let $U \subseteq \mathbb{R}^4$ be the following 2-plane in $\mathbb{R}^4$:

$$U = s\begin{pmatrix}1\\1\\2\\2\end{pmatrix} + t\begin{pmatrix}1\\2\\3\\4\end{pmatrix} = s\vec{u} + t\vec{v}$$

We want to find all the vectors $\vec{x} \in \mathbb{R}^4$ that are perpendicular to this 2-plane.

[ This should remind you of the cross product, but the cross product only works in $\mathbb{R}^3$. ]

In our case:

$$U^\perp = \{\vec{x} \in \mathbb{R}^4 : \vec{u} \cdot \vec{x} = 0 \ \& \ \vec{v} \cdot \vec{x} = 0\}$$

In other words, $U^\perp$ is the solution set of the following linear system:

$$\begin{cases} x_1 + x_2 + 2x_3 + 2x_4 = 0, \\ x_1 + 2x_2 + 3x_3 + 4x_4 = 0. \end{cases}$$

$$\Big\downarrow \text{RREF}$$

$$\begin{cases} x_1 + 0 + x_3 + 0 = 0, \\ 0 + x_2 + x_3 + 2x_4 = 0. \end{cases}$$

Pivot variables: $x_1, x_2$

Free variables: $x_3, x_4$.

Rename: $s = x_3$, $t = x_4$

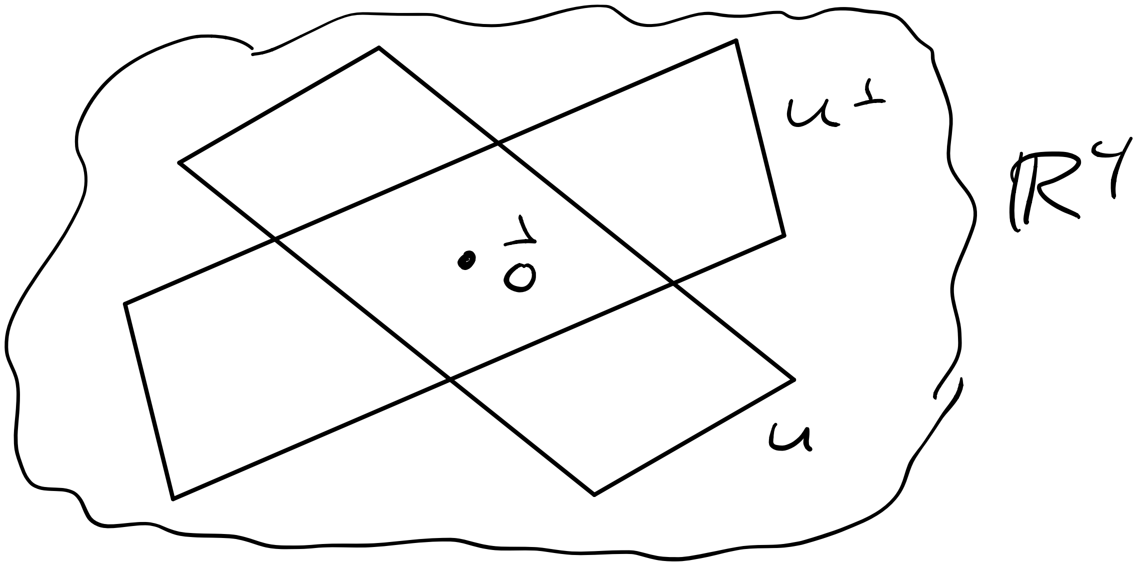
Solution:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -s \\ -s-2t \\ s \\ t \end{pmatrix}$$

$$= s\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ -2 \\ 0 \\ 1 \end{pmatrix}.$$

This is a 2-plane in $\mathbb{R}^4$.

So, the orthogonal complement of a 2-plane in $\mathbb{R}^4$ is another 2-plane. Bad Picture:

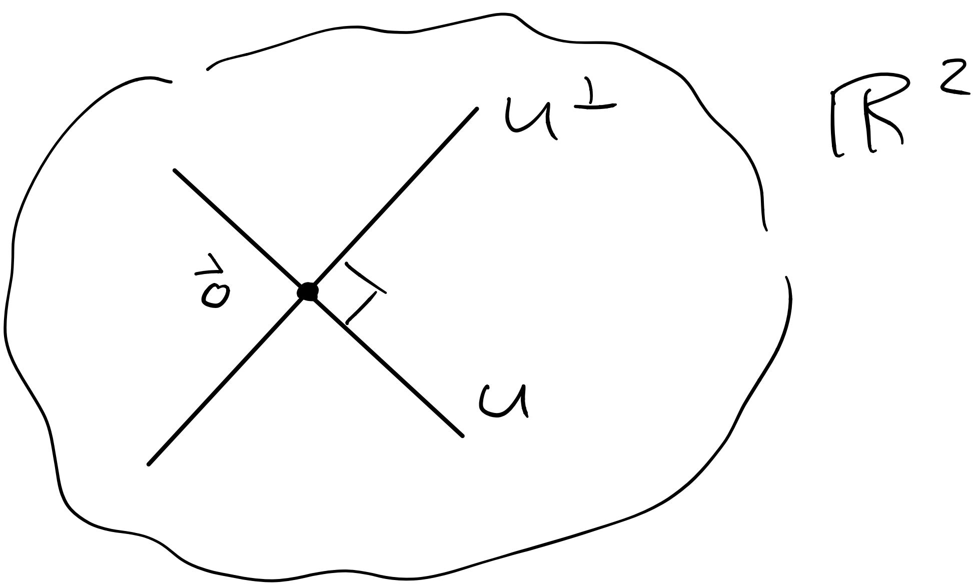


We have two 2-planes in $\mathbb{R}^4$ that are perpendicular and meet only at a single point, the origin.

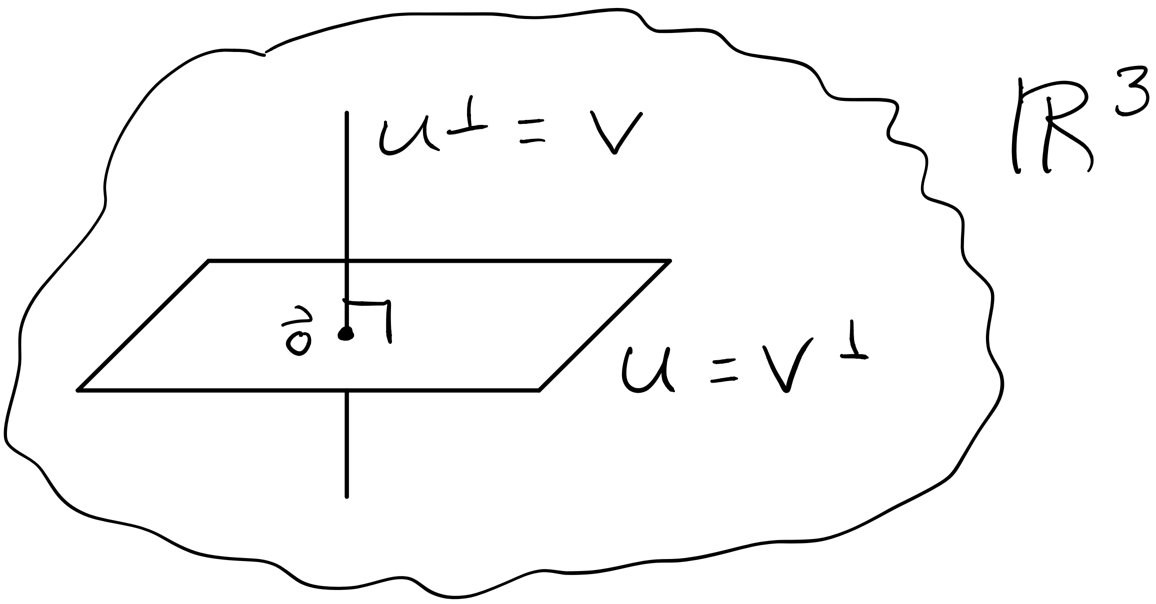
Can you visualize this ?

No you can't !

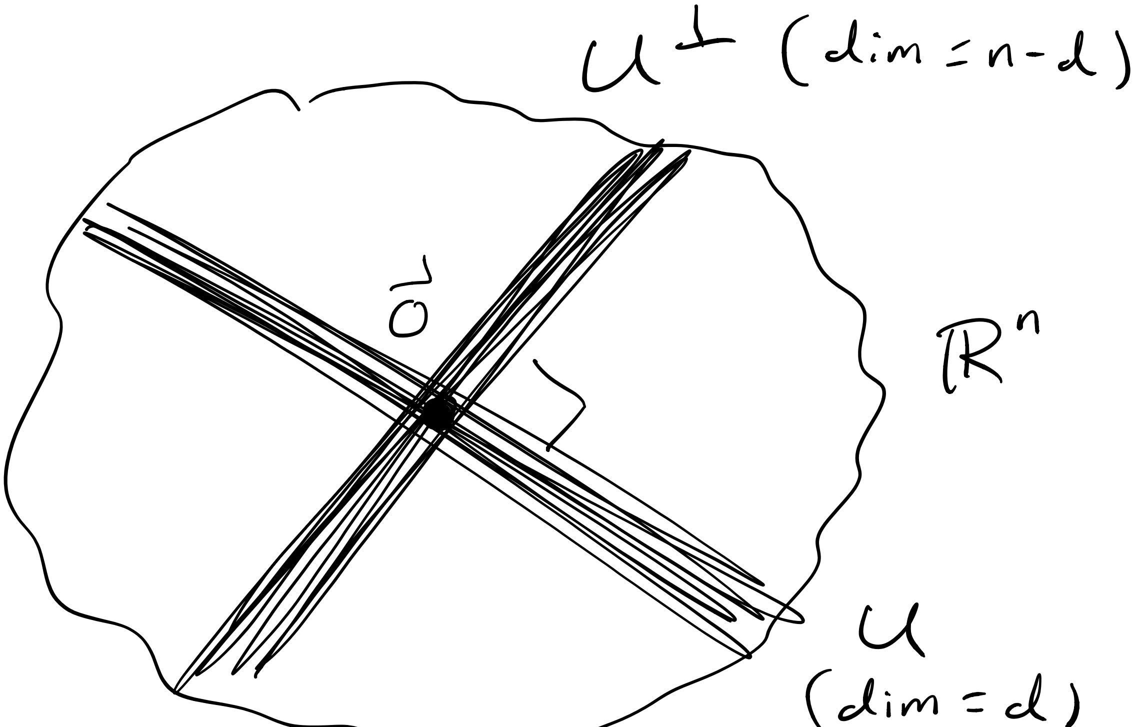
The only examples that can be visualized are in $\mathbb{R}^2$, where orthogonal subspaces are pairs of perpendicular lines :

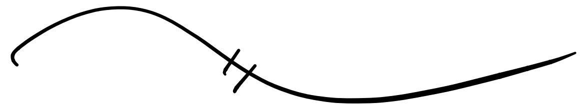



Or in $\mathbb{R}^3$, where orthogonal subspaces are perpendicular (line, plane) pairs :




The general situation can only be hinted at with a diagram such as this:





Topics for Quiz 2:

- Computing RREF of any matrix.

- Interpreting the RREF of a linear system:

  $m$ = # equations
  $n$ = # unknowns
  $r$ = # pivot variables in RREF
  $d$ = # free variables in RREF

Then

$$0 \le r \le m$$
$$r + d = n$$
$$n - m \le d \le n.$$

The solution set is either empty or it is a $d$-plane in $\mathbb{R}^n$.