HW5 due right now.

Remainder of the Course:

1. Least Squares Approximation ✓
2. Diagonalization, i.e., finding the "correct" coordinate system for your problem. (Next)

Quiz 5: Tuesday Nov 10.

Today: HW5 Discussion, Odds & Ends.

Problem 1: Write $c = \cos\theta$, $s = \sin\theta$.

$$R_\theta = \begin{pmatrix} c & -s \\ s & c \end{pmatrix}, \quad F_\theta = \begin{pmatrix} c & s \\ s & -c \end{pmatrix}, \quad P_\theta = \begin{pmatrix} c^2 & cs \\ cs & s^2 \end{pmatrix}.$$

Determinants:

$$\left.\begin{aligned} \det(R_\theta) &= c^2 + s^2 = 1 \\ \det(F_\theta) &= -c^2 - s^2 = -1 \\ \det(P_\theta) &= c^2 s^2 - c^2 s^2 = 0 \end{aligned}\right\} \text{ } \theta \text{ doesn't matter.}$$

Recall: 2×2 matrix A is invertible

$\iff$ $A\vec{x} = \vec{0} \Rightarrow \vec{x} = \vec{0}$

$\iff$ columns of A not parallel

$\iff$ rows of A not parallel

$\iff$ $\det(A) \neq 0$.

In which case we can write an explicit formula for the inverse:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

So in our case, R_θ & F_θ are invertible with

- $R_\theta^{-1} = \frac{1}{1} \begin{pmatrix} c & s \\ -s & c \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

Observe: $R_\theta^{-1} = R_{-\theta}$

(rotate by θ)$^{-1}$ = (rotate by $-\theta$)

- $F_\theta^{-1} = \frac{1}{-1}\begin{pmatrix} -c & -s \\ -s & c \end{pmatrix}$

$$= \begin{pmatrix} c & s \\ s & -c \end{pmatrix} = F_\theta \;!$$

$$(\text{reflect})^{-1} = (\text{same reflection})$$

$$F_\theta^{-1} = F_\theta$$

$$\cancel{F_\theta^{-1}} F_\theta = F_\theta F_\theta$$

$$I = F_\theta^2$$

$$(\text{do nothing}) = (\text{reflect twice})$$

The matrix $P_\theta = \begin{pmatrix} c^2 & cs \\ cs & s^2 \end{pmatrix}$ is

not invertible because $\det(P_\theta) = 0$.

What does this matrix do?

Claim: P_θ is the projection onto the line $t(\cos\theta, \sin\theta)$, i.e., the line of angle θ.

Proof: Let $\vec{a} = (\cos\theta, \sin\theta)$ so the projection matrix is

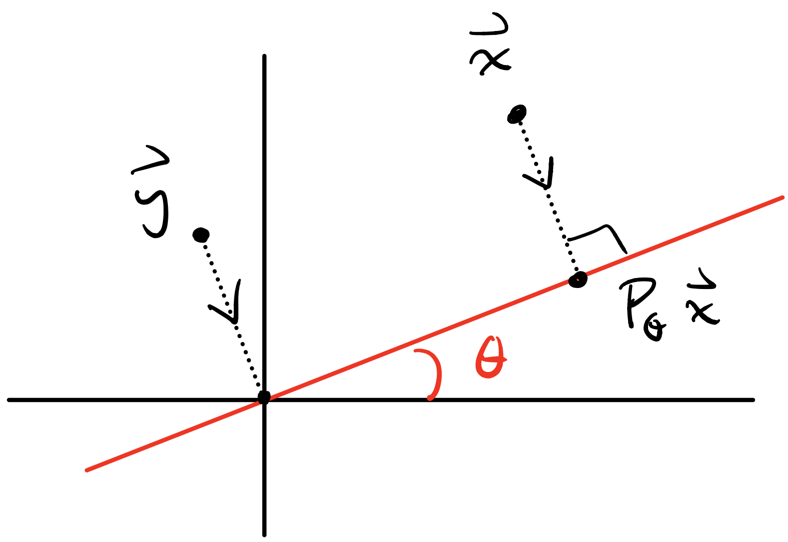
$$P = \vec{a}(\vec{a}^T\vec{a})^{-1}\vec{a}^T$$

$$= \begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix}\Big[\underbrace{\cancel{(\cos\theta \ \ \sin\theta)\begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix}}}_{\underbrace{\cancel{\cos^2\theta + \sin^2\theta}}_{1}}\Big]^{-1}(\cos\theta \ \ \sin\theta)$$

$$= \begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix}(\cos\theta \ \ \sin\theta)$$

$$= \begin{pmatrix}\cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta\end{pmatrix} \checkmark$$

Picture:



Another reason why P_θ is not invertible: It send some nonzero vectors to zero.

Problem 2: The <u>algebra</u> of projection matrices. Say P is a "projection matrix" if

$$P^T = P \quad \& \quad P^2 = P.$$

(a) Let $Q = I - P$. Then Q is also a projection matrix:

- $Q^T = (I-P)^T$

$$= I^T - P^T = I - P = Q \checkmark$$

[Jargon: Matrices $A^T = A$ are called <u>symmetric</u>. The entries are symmetric across the main diagonal.]

$$\bullet \quad Q^2 = (I-P)(I-P)$$
$$= I^2 - IP - PI + PP$$
$$= I - P - \cancel{P} + \cancel{P}$$
$$= I - P = Q \quad \checkmark$$

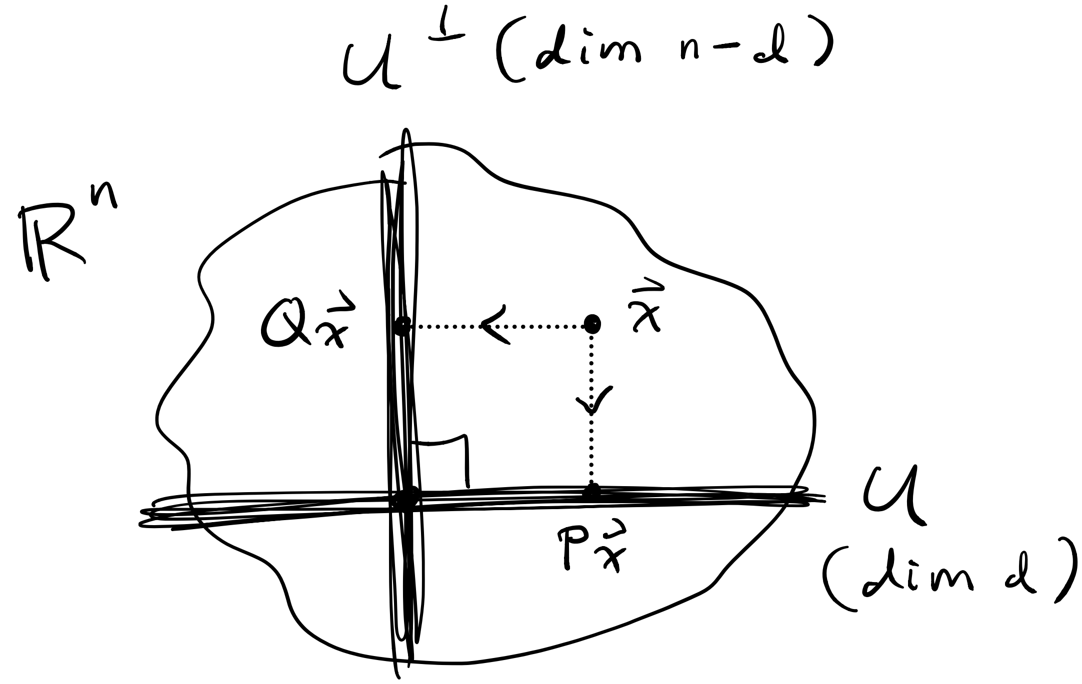
(b) $$PQ = P(I-P)$$
$$= P - P^2$$
$$= P - P = 0.$$

Summary of (a) & (b): Every projection P comes with a "complementary projection" Q satisfying:

$$P + Q = I \quad \& \quad PQ = QP = 0.$$

What is the picture?

P & Q project onto a pair of "complementary subspaces":



[Projections, like subspaces, come in complementary pairs.]

(c) It is "fun" to check that

$P = A(A^TA)^{-1}A^T$ satisfy

$$P^T = P \text{ \& } P^2 = P.$$

[See the solutions.]

Geometrically, this is the matrix that projects onto the column space of matrix A.

(d) What if A is square & invertible?

Then we get

$$P = A(A^TA)^{-1}A^T$$

$$= \cancel{A(A)^{-1}}\,\cancel{(A^T)^{-1}A^T}$$

$$= I\,I = I$$

"(project onto $C(A)$) = (do nothing)"

What?

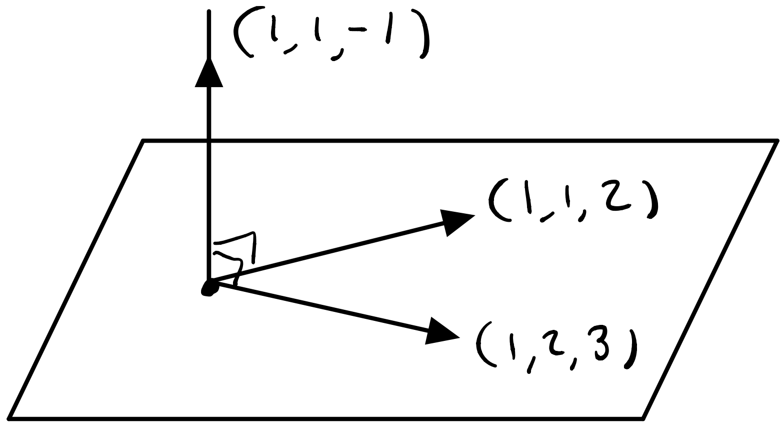
If A is invertible then $C(A)$ is the whole space. Since every point is already in the whole space, there is no need to do anything!

Example: Problem 3.

Let $\vec{a} = (1,1,-1)^T$

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 2 \end{pmatrix}.$$

Observe that line $C(\vec{a})$ & plane $C(A)$ are "ortogonal complements":



If P = project onto line,
Q = project onto plane,

then we will have

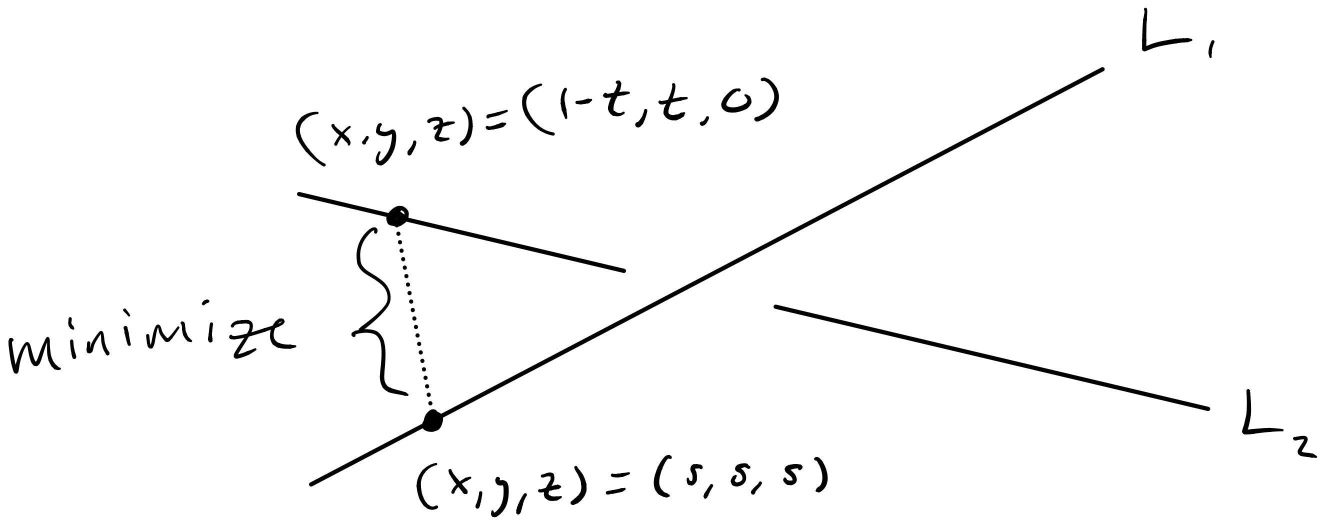
$$P + Q = I \quad \& \quad PQ = QP = 0.$$

I asked you to compute the matrices and check that this indeed the case.

[See solutions.]

Problem 4 : Geometric Application of Least Squares.

We have two lines in $\mathbb{R}^3$ that do not intersect:



Find the points on L_1 & L_2 that minimize the distance.

Remark: We could use calulus. Let

$f(s,t)$ = distance between (s,s,s) & $(1-t,t,0)$

$$= \sqrt{(s-1+t)^2 + (s-t)^2 + s^2}$$

Then compute first & second partial derivatives, etc...

But we will use linear algebra!

Idea: Write down a silly equation.

The lines L_1 & L_2 intersect when

$$(s, s, s) = (1-t, t, 0)$$

$$\begin{cases} s = 1-t \\ s = t \\ s = 0 \end{cases} \rightsquigarrow \begin{cases} s+t = 1 \\ s-t = 0 \\ s+0 = 0 \end{cases}$$

Of course there is no solution, so we solve the "normal equations" instead:

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \hat{s} \\ \hat{t} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} \hat{s} \\ \hat{t} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\Longrightarrow \quad \hat{s} = \frac{1}{3} \quad \& \quad \hat{t} = \frac{1}{2}.$$

Much easier than calculus!

Solution: Distance between L_1 & L_2 is minimized for the points

$$(\hat{s}, \hat{s}, \hat{s}) = \left(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right) \text{ on } L_1$$

$$(1-\hat{t}, \hat{t}, 0) = \left(\tfrac{1}{2}, \tfrac{1}{2}, 0\right) \text{ on } L_2$$

Minimum distance between L_1 & L_2:

$$\sqrt{\left(\tfrac{1}{2}-\tfrac{1}{3}\right)^2 + \left(\tfrac{1}{2}-\tfrac{1}{3}\right)^2 + \left(-\tfrac{1}{3}\right)^2}$$

$$= \sqrt{\left(\tfrac{1}{36} + \tfrac{1}{36} + \tfrac{1}{9}\right)} = 1/\sqrt{6}$$

Problem 5: The usual application of least squares: Fitting shapes to collections of data points.

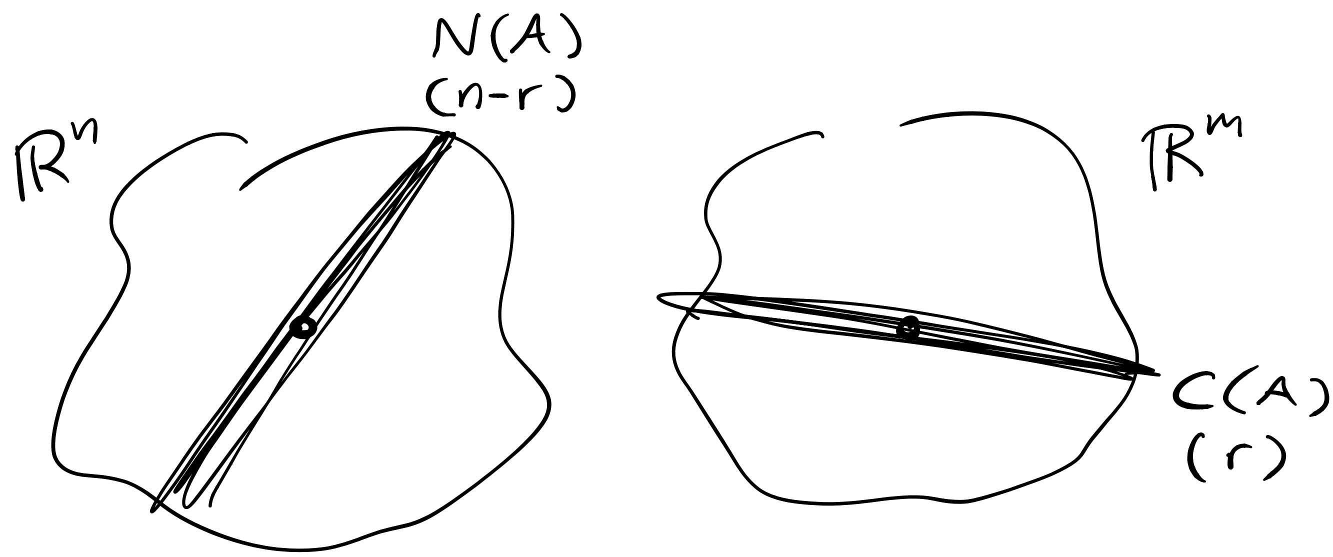
[See solutions.]

Odds & Ends: The 4 fundamental subspaces & the fundamental theorem.

Given $m \times n$ matrix A we define

$$C(A) = \text{column space} = \{ A\vec{x} : \vec{x} \in \mathbb{R}^n \} \subseteq \mathbb{R}^m$$

$$N(A) = \text{nullspace} = \{ \vec{x} \in \mathbb{R}^n : A\vec{x} = \vec{0} \} \subseteq \mathbb{R}^n.$$



Let $r = \text{rank}(A)$

$= \#$ pivots in RREF(A)

Then $\dim C(A) = r$

$\dim N(A) = n - r$

$= \#$ non-pivot columns in RREF(A)

Two more fundamental subspaces

$R(A)$ = row space of A

= column space of A^T

$= C(A^T)$

$N(A^T) = \{ \vec{x} \in \mathbb{R}^m : A^T \vec{x} = \vec{0} \}$

Recall from least squares:

$$N(A^T) = C(A)^\perp$$

Since these are subspaces of $\mathbb{R}^m$ we know from HW3.5 that

$$\dim N(A^T) + \dim C(A) = m$$

$$\dim N(A^T) = m - \dim C(A)$$

$$= m - r.$$

And by plugging $B = A^T$ into the previous formula we get

$$N(B) = C(B^T)^\perp$$

$$N(B) = R(B)^\perp$$

For any B. In particular,

$$N(A) = R(A)^\perp$$

Since these are subspaces of $\mathbb{R}^n$ we know from HW 3.5 that

$$\dim R(A) + \dim N(A) = n$$

$$\begin{aligned} \dim R(A) &= n - \dim N(A) \\ &= n - (n-r) \\ &= r. \end{aligned}$$

Surprise!

Let me state this as an official theorem.

The Fundamental Theorem of Linear Algebra:

For any matrix A we have

$$\operatorname{rank}(A) = \operatorname{rank}(A^T)$$

$$\dim C(A) = \dim R(A)$$

$$\#\text{pivots in } RREF(A) = \#\text{pivots in } RREF(A^T)$$

This is a very simple statement, but as you see above, it was not easy to prove. Example:

Let $A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 1 & 3 & 0 \end{pmatrix}$ & $A^T = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 1 & 0 \end{pmatrix}$

$$RREF(A) = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$RREF(A^T) = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

So $\operatorname{rank}(A) = \operatorname{rank}(A^T) = 2$.

We compute the 4 fundamental subspaces of A:

Columns 1 & 2 of A are pivots, so

$$C(A) = \text{plane } s\begin{pmatrix}1\\1\\1\end{pmatrix} + t\begin{pmatrix}1\\2\\3\end{pmatrix}.$$

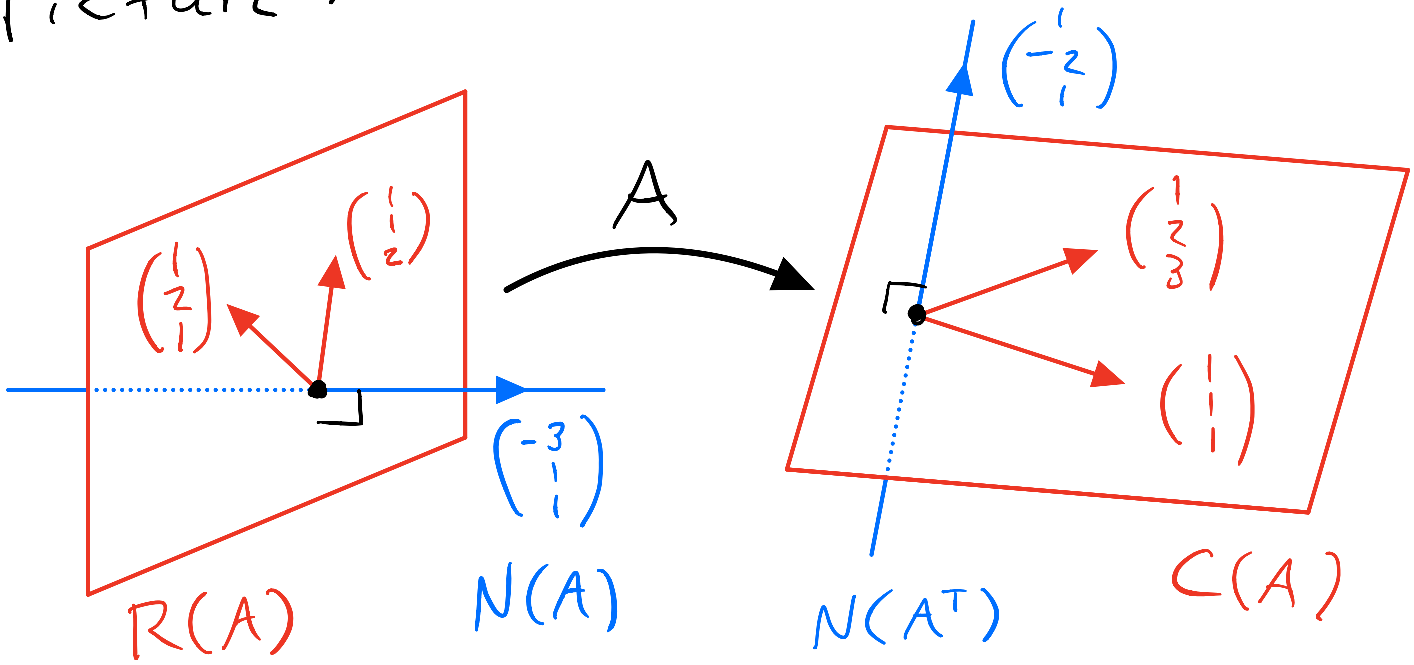
$$N(A^T) = C(A)^\perp = \text{line } t\begin{pmatrix}1\\-2\\1\end{pmatrix}$$

Columns 1 & 2 of A^T are pivots, so

$$R(A) = C(A^T) = \text{plane } s\begin{pmatrix}1\\1\\2\end{pmatrix} + t\begin{pmatrix}1\\2\\1\end{pmatrix}.$$

$$N(A) = R(A)^\perp = \text{line } t\begin{pmatrix}-3\\1\\1\end{pmatrix}.$$

Picture:



Don't worry too much about the fundamental theorem; I just wanted you to see it.

Quiz 5 is on Tuesday at the beginning of class. As usual there will be two problems:

- one problem about projection
- one problem about least squares