HW6 due Thurs before class.

Final Project:
Minimum 2 pages, single spaced.
Give a point form summary of what you learned in the class. Basically it's a chance for you to review the course material.

Current Topic: Diagonalization, i.e., Eigenvalues & Eigenvectors.

Idea: Given a square matrix A, the eigenvectors of A are the "correct coordinate system" to analyze the matrix.

Technically: We say that scalar λ is an eigenvalue of A if there exists a nonzero vector $\vec{u} \neq \vec{0}$

such that $A\vec{u} = \lambda\vec{u}$. In this case we say that $\vec{u}$ is a λ-eigenvector of A.

Observe: If A is $n \times n$, then the λ-eigenvectors of A form a subspace of $\mathbb{R}^n$. Indeed,

$$
\begin{aligned}
& A\vec{u} = \lambda\vec{u} \\
\iff\ & A\vec{u} = \lambda I\vec{u} \\
\iff\ & A\vec{u} - \lambda I\vec{u} = \vec{0} \\
\iff\ & (A - \lambda I)\vec{u} = \vec{0}
\end{aligned}
$$

So the λ-eigenvectors of A are the NONZERO solutions of a linear system, hence they form a subspace of $\mathbb{R}^n$.

Jargon:

$$E_\lambda = \text{nullspace of } A - \lambda I$$

$$= N(A-\lambda I)$$

$$= \{ \vec{u} : (A-\lambda I)\vec{u} = \vec{0} \}$$

is called the λ-eigenspace of A.

Examples:

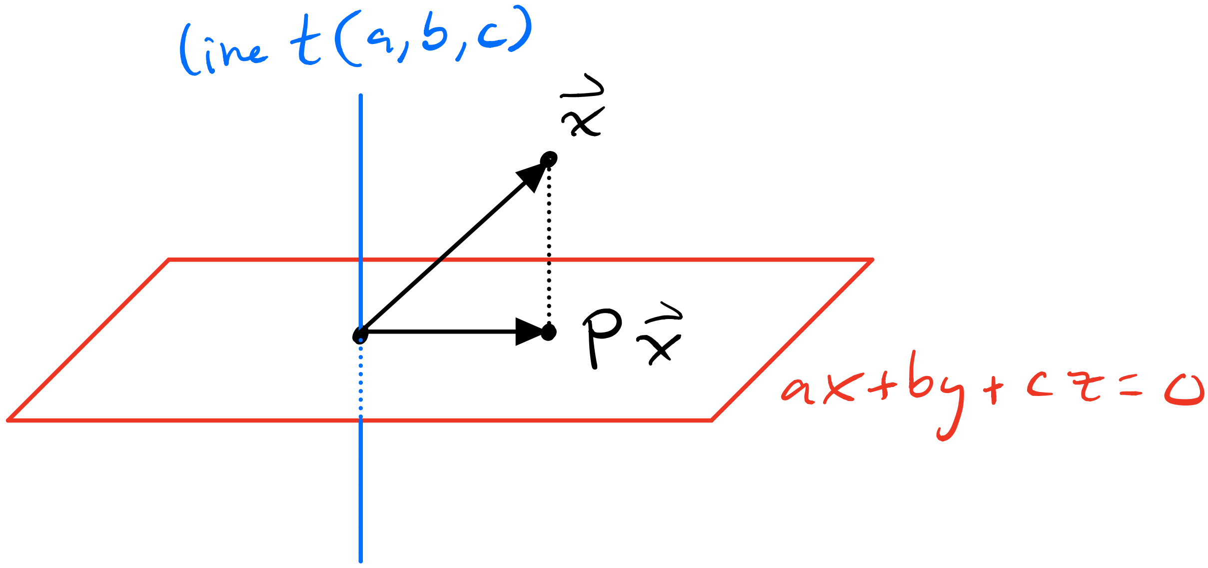
- Let A = $n \times n$ zero matrix O.
  Since $A\vec{u} = \vec{0} = 0\vec{u}$ for all $\vec{u}$, we see that 0 is the only eigenvalue of A. The 0-e.space is the whole space:

$$E_0 = \mathbb{R}^n.$$

- Let A = $n \times n$ identity matrix I, so $A\vec{u} = \vec{u} = 1\vec{u}$ for all $\vec{u}$. Thus 1 is the only eigenvalue, and

$$E_1 = \mathbb{R}^n.$$

• Let P = 3×3 matrix that projects (orthogonally) onto the plane $ax+by+cz=0$:



For which $\vec{x}$ do we have $P\vec{x}=\lambda\vec{x}$?

When do $\vec{x}$ & $P\vec{x}$ point in "same direction"?

Claim: The eigenvalues of P are 1 and 0. The eigenspaces are:

E_1 = plane $ax+by+cz=0$,

E_0 = line $t(a,b,c)$.

Check: If $\vec{x}$ is <u>in the plane</u> then

$$P\vec{x}=\vec{x}=1\vec{x} \quad \checkmark$$

If $\vec{x}$ is on the line $t(a,b,c)$ then it gets projected to the origin:

$$P\vec{x} = \vec{0} = 0\vec{x} \quad \checkmark$$

There are no other eigenvectors!

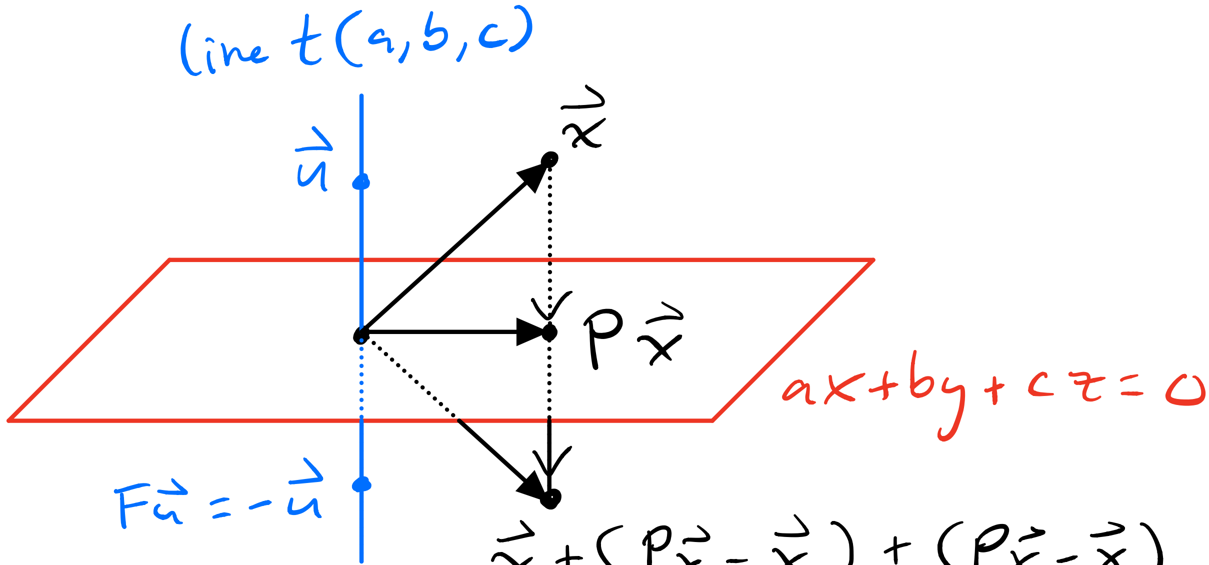
- Now let $F = 2P - I$. I claim that F & P have <u>the same eigenspaces</u>; but the eigenvalues change. See:

$$\begin{aligned} P\vec{u} &= \lambda\vec{u} \\ F\vec{u} &= (2P - I)\vec{u} \\ &= 2P\vec{u} - I\vec{u} \\ &= 2\lambda\vec{u} - \vec{u} \\ &= (2\lambda - 1)\vec{u}. \end{aligned}$$

If $\vec{u}$ is λ-eigenvector of P then $\vec{u}$ is a $(2\lambda-1)$-eigenvector of F.

Since P has eigenvalues $\lambda = 1$ & 0.
F has eigenvalues $2\lambda - 1 = 1$ & -1.

Picture:



$$\begin{aligned} &\vec{x} + (P\vec{x} - \vec{x}) + (P\vec{x} - \vec{x}) \\ &= 2P\vec{x} - \vec{x} \\ &= (2P - I)\vec{x} \\ &= F\vec{x}. \end{aligned}$$

So F is the <u>reflection</u> across the plane $ax+by+cz=0$.

E_1 = plane $ax+by+cz=0$,

E_{-1} = line $t(a,b,c)$.

General fact: Consider a "polynomial"

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n.$$

If A is a square matrix then we can "evaluate f at A" to get a matrix:

$$f(A) := a_0 I + a_1 A + a_2 A^2 + \cdots + a_n A^n$$

If $A\vec{u} = \lambda \vec{u}$, then one can check that

$$f(A)\vec{u} = f(\lambda)\vec{u}$$

so that $\vec{u}$ is an eigenvector of the matrix $f(A)$ with eigenvalue $f(\lambda)$.

If we don't know the geometry of A then we have to do computations.

Recall: λ is an eigenvalue of A

$$\iff E_\lambda = N(A - \lambda I) \neq \{\vec{0}\}.$$

$\iff (A-\lambda I)^{-1}$ does not exist

$\iff \det(A-\lambda I) \neq 0$

The "characteristic equation" of A.

Example: $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$.

Find the eigenvalues:

$$\det(A-\lambda I) = 0$$

$$\det\begin{pmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{pmatrix} = 0$$

$$(1-\lambda)(3-\lambda) - 2\cdot 4 = 0$$

$$\lambda^2 - 4\lambda + 3 - 8 = 0$$

$$\lambda^2 - 4\lambda - 5 = 0$$

$$(\lambda+1)(\lambda-5) = 0$$

[or use quadratic formula]

Eigenvalues: $\lambda = -1$ or 5.

Now we find the eigenspaces:

$$E_{-1} = N(A-(-1)I)$$

$$= N\begin{pmatrix} 2 & 2 \\ 4 & 4 \end{pmatrix}$$

$$= \left\{ \vec{u} : \begin{pmatrix} 2 & 2 \\ 4 & 4 \end{pmatrix} \vec{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

$$\left(\begin{array}{cc|c} 2 & 2 & 0 \\ 4 & 4 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$\vec{u} = t\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

E_{-1} = the line $t\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$E_5 = N(A-5I)$$

$$= N\begin{pmatrix} -4 & 2 \\ 4 & -2 \end{pmatrix}$$

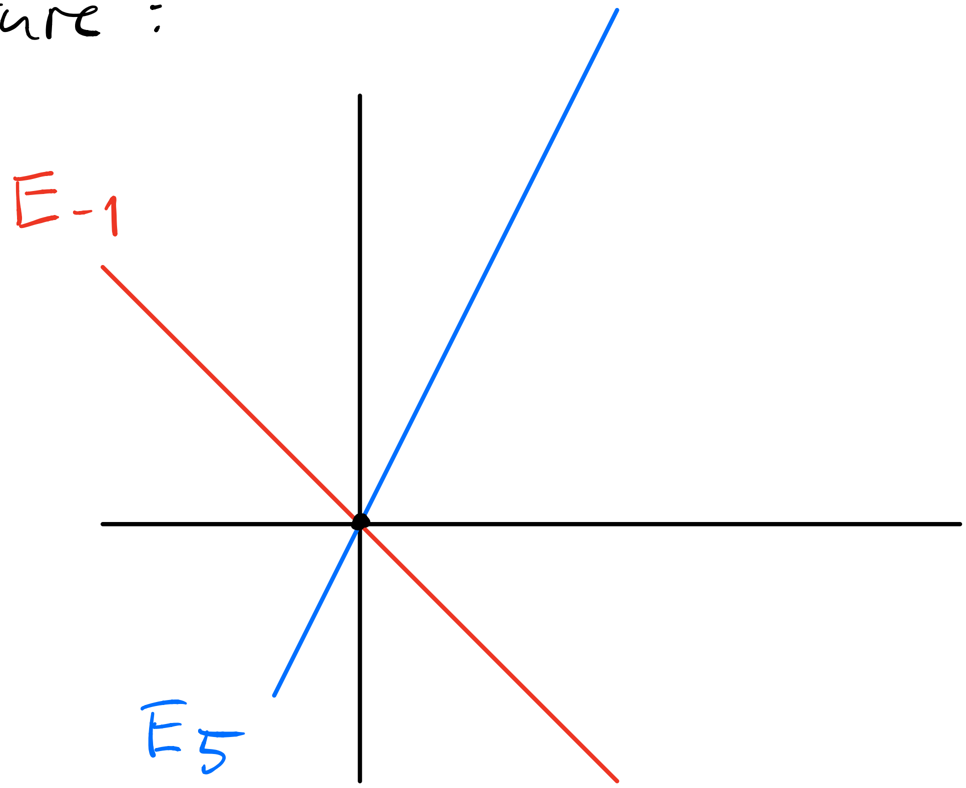
$$= \left\{ \vec{v} : \begin{pmatrix} -4 & 2 \\ 4 & -2 \end{pmatrix} \vec{v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

$$\left(\begin{array}{cc|c} -4 & 2 & 0 \\ 4 & -2 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$\vec{v} = t\begin{pmatrix}1\\2\end{pmatrix}$

$E_5 = $ line $t\begin{pmatrix}1\\2\end{pmatrix}$.

Picture:



These lines tell us the "correct coordinate system" to analyze A.

Example: Consider the recurrence

$\vec{x}_0 = \begin{pmatrix}5\\4\end{pmatrix}$ & $\vec{x}_{n+1} = A\vec{x}_n$.

Solution: Express initial condition $\vec{x}_0 = (5,4)$ in terms of the eigenvectors,

$$a\begin{pmatrix}1\\-1\end{pmatrix} + b\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}5\\4\end{pmatrix}$$

$$\begin{pmatrix}1 & 1\\-1 & 2\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}5\\4\end{pmatrix}$$

$$\begin{pmatrix}a\\b\end{pmatrix} = \frac{1}{3}\begin{pmatrix}2 & -1\\1 & 1\end{pmatrix}\begin{pmatrix}5\\4\end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix}6\\9\end{pmatrix} = \begin{pmatrix}2\\3\end{pmatrix}$$

$$\Longrightarrow \vec{x}_0 = \begin{pmatrix}5\\4\end{pmatrix} = 2\begin{pmatrix}1\\-1\end{pmatrix} + 3\begin{pmatrix}1\\2\end{pmatrix}.$$

Now we just write down the solution:

$$\vec{x}_n = AA\cdots A\vec{x}_0$$

$$= A^n\vec{x}_0$$

$$= A^n\left[2\begin{pmatrix}1\\-1\end{pmatrix} + 3\begin{pmatrix}1\\2\end{pmatrix}\right]$$

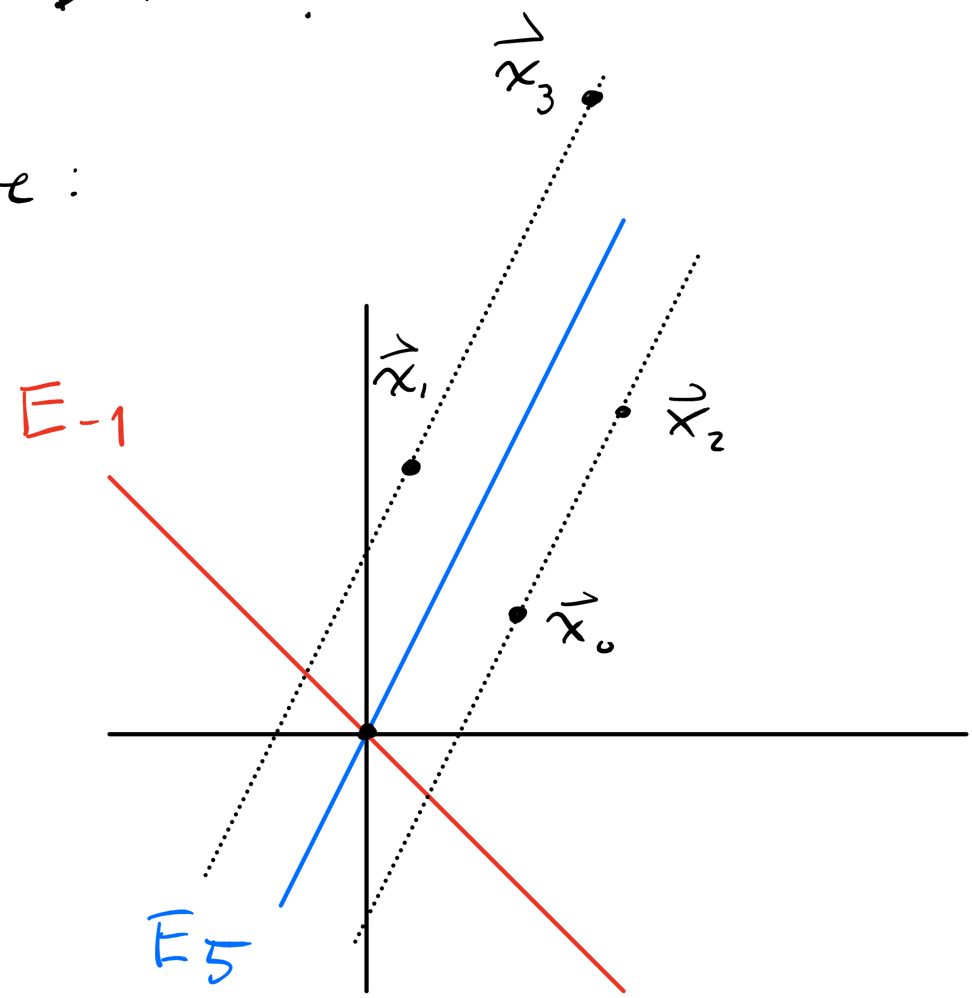
$$= 2A^n\begin{pmatrix}1\\-1\end{pmatrix} + 3A^n\begin{pmatrix}1\\2\end{pmatrix}$$

$$= 2(-1)^n \begin{pmatrix} 1 \\ -1 \end{pmatrix} + 3 \cdot 5^n \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2(-1)^n + 3(5)^n \\ -2(-1)^n + 6(5)^n \end{pmatrix}.$$

DONE !

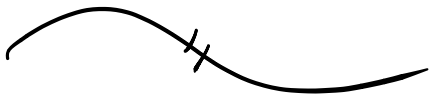
Picture :



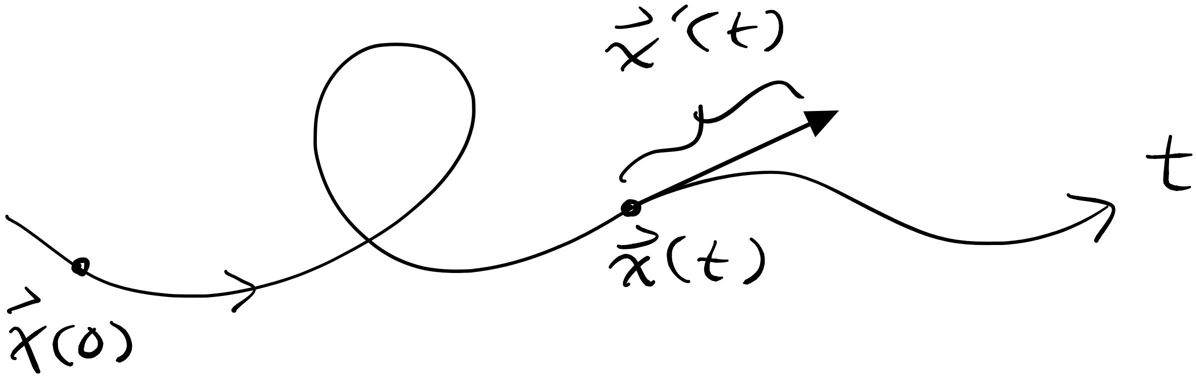
At every step : Red coordinate is multiplied by -1 & blue coordinate is multiplied by 5.

This kind of problem is called

a "discrete dynamical system."



Final Topic: Continuous dynamical systems. A function $\mathbb{R} \to \mathbb{R}^n$ sending scalar t to point $\vec{x}(t)$ is called a "parametrized path" in $\mathbb{R}^n$:



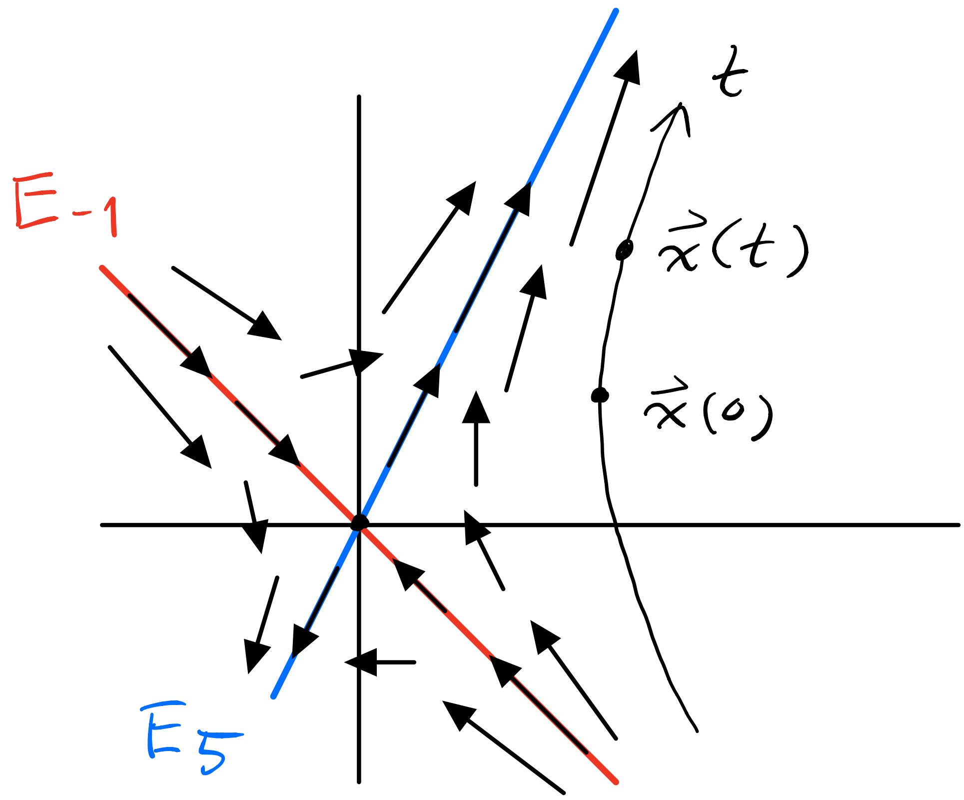
If $\vec{x}(t) = (x(t), y(t))$ are the coordinates of the position at time t, then the velocity vector at time t is

$$\vec{x}'(t) = (x'(t), y'(t))$$

$$= \left(\frac{dx}{dt}, \frac{dy}{dt}\right).$$

An $n \times n$ matrix A can be viewed as a "velocity vector field." At each point $\vec{x}$ we have a vector $A\vec{x}$.

Example: $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$ determines the following vector field:



Given an initial position $\vec{x}(0)$ our goal is to find the path $\vec{x}(t)$ that follows this vector field. In other words, the velocity vector should satisfy

$$\vec{x}'(t) = A\vec{x}(t)$$

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

$$\begin{cases} x'(t) = x(t) + 2y(t) \\ y'(t) = 4x(t) + 3y(t) \end{cases}$$

This is a system of two linear "differential equations."

I claim that we already did all the work to solve this problem!

Theorem: Suppose A has eigenvectors

$$A\vec{u} = \lambda\vec{u},$$
$$A\vec{v} = \mu\vec{v}.$$

Then the solution of the differential equation $\vec{x}'(t) = A\vec{x}(t)$ with

initial position $\vec{x}(0) = a\vec{u} + b\vec{v}$
is given by

$$\vec{x}(t) = a e^{\lambda t} \vec{u} + b e^{\mu t} \vec{v}$$

In our example, let the initial position be

$$\vec{x}(0) = \begin{pmatrix} 5 \\ 4 \end{pmatrix} = 2\begin{pmatrix} 1 \\ -1 \end{pmatrix} + 3\begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Then the solution is

$$\vec{x}(t) = 2e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + 3e^{5t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 2e^{-t} + 3e^{5t} \\ -2e^{-t} + 6e^{5t} \end{pmatrix}.$$

You can check that this is correct by differentiating $x(t)$ & $y(t)$.
Next time we'll discuss WHY it works.