HW 6 will be posted later today, due next Thurs Nov 19 before class.

There is no Quiz 6.

Final Project due Tues Dec 1.

Final Topic of the course:

"Diagonalization"

or

"Spectral Analysis"

Basic Idea: Given a square matrix A we want to find the "correct" coordinate system to work with this matrix. The correct coordinate vectors are called <u>eigenvectors</u> of A.

Last time we considered the following "discrete dynamical system":

$$\begin{cases} x_{n+1} = 0.4x_n + 0.2y_n, \\ y_{n+1} = 0.6x_n + 0.8y_n. \end{cases}$$

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 0.4 & 0.2 \\ 0.6 & 0.8 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

$$\vec{x}_{n+1} = A\vec{x}_n$$

"linear recurrence equation"

The key to solving this system is the following two mysterious facts:

- $A\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$
- $A\begin{pmatrix} -1 \\ 1 \end{pmatrix} = 0.2\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Once we know this, the solution is easy.

Suppose we start with

$$\vec{x}_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 100 \\ 0 \end{pmatrix}.$$

Express $\vec{x}_0$ in terms of $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ & $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$:

$$\vec{x}_0 = \begin{pmatrix} 100 \\ 0 \end{pmatrix} = 25\begin{pmatrix} 1 \\ 3 \end{pmatrix} - 75\begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Final Solution:

$$\begin{aligned}
\vec{x}_n &= AAA\cdots A\vec{x}_0 \\
&= A^n\vec{x}_0 \\
&= A^n\left[25\begin{pmatrix} 1 \\ 3 \end{pmatrix} - 75\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right] \\
&= 25A^n\begin{pmatrix} 1 \\ 3 \end{pmatrix} - 75A^n\begin{pmatrix} -1 \\ 1 \end{pmatrix} \\
&= 25\begin{pmatrix} 1 \\ 3 \end{pmatrix} - 75(0.2)^n\begin{pmatrix} -1 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 25 + 75(0.2)^n \\ 75 - 75(0.2)^n \end{pmatrix}.
\end{aligned}$$

We get exact formulas for

$$x_n = 25 + 75(0.2)^n$$
$$y_n = 75 - 75(0.2)^n.$$

That's pretty good!

Let's go one step further. Define

$$U = \begin{pmatrix} 1 & -1 \\ 3 & 1 \end{pmatrix} \;\&\; \Lambda = \begin{pmatrix} 1 & 0 \\ 0 & 0.2 \end{pmatrix}.$$

Then I claim that

$$AU = U\Lambda.$$

Check:

$$AU = A\begin{pmatrix} 1 & -1 \\ 3 & 1 \end{pmatrix}$$

$$= \left(A\begin{pmatrix} 1 \\ 3 \end{pmatrix} \; A\begin{pmatrix} -1 \\ 1 \end{pmatrix} \right)$$

$$= \begin{pmatrix} 1 & -0.2 \\ 3 & 0.2 \end{pmatrix}.$$

$$U\Lambda = \begin{pmatrix} 1 & -1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0.2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -0.2 \\ 3 & 0.2 \end{pmatrix} \quad ✓$$

We can use this equation to compute the nth power of matrix A. Since U is invertible:

$$A = U \Lambda U^{-1}$$

we have "diagonalized" A.

$$A^2 = U \Lambda \cancel{U^{-1} U} \Lambda U^{-1}$$
$$= U \Lambda \Lambda U^{-1}$$
$$= U \Lambda^2 U^{-1}$$

$$A^3 = U \Lambda^3 U^{-1}$$
$$\vdots$$
$$A^n = U \Lambda^n U^{-1}.$$

This is great because computing Λ^n is easy:

$$\Lambda^n = \begin{pmatrix} 1 & 0 \\ 0 & 0.2 \end{pmatrix}^n$$
$$= \begin{pmatrix} 1^n & 0 \\ 0 & (0.2)^n \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & (0.2)^n \end{pmatrix}$$

["Diagonal matrices are easy to work with."]

we conclude that

$$\begin{pmatrix} 0.4 & 0.2 \\ 0.6 & 0.8 \end{pmatrix}^n = A^n$$

$$= U \Lambda^n U^{-1}$$

$$= \begin{pmatrix} 1 & -1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & (0.2)^n \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 3 & 1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 1 & -(0.2)^n \\ 3 & (0.2)^n \end{pmatrix} \frac{1}{4} \begin{pmatrix} 1 & 1 \\ -3 & 1 \end{pmatrix}$$

$$= \frac{1}{4} \begin{pmatrix} 1+3(0.2)^n & 1-(0.2)^n \\ 3-3(0.2)^n & 3+(0.2)^n \end{pmatrix}.$$

That's pretty good!

So for any initial conditions x_0, y_0:

$$\vec{x}_n = A \vec{x}_0$$

$$= \frac{1}{4} \begin{pmatrix} 1+3(0.2)^n & 1-(0.2)^n \\ 3-3(0.2)^n & 3+(0.2)^n \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

= whatever this is.

As $n \to \infty$ we see that

$$A^n \to \frac{1}{4}\begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix}$$

$$\vec{x}_n \to \frac{1}{4}\begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$$= \begin{pmatrix} (x_0+y_0)/4 \\ 3(x_0+y_0)/4 \end{pmatrix}.$$

[In the far future, we have
- 1/4 bears in the valley
- 3/4 bears on the mountain.]

How did I find the key equations

$$A\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad \& \quad A\begin{pmatrix} -1 \\ 1 \end{pmatrix} = 0.2\begin{pmatrix} -1 \\ 1 \end{pmatrix} \ ?$$

Definition: Let A be a square matrix.

A scalar λ is called an eigenvalue of A

if there exists a nonzero vector $\vec{u} \neq \vec{0}$ such that

$$\boxed{A\vec{u} = \lambda\vec{u}}$$

In this case $\vec{u}$ is called a λ-eigenvector of A.

[Warning: The zero vector $\vec{0}$ is not allowed to be an eigenvector! Because the equation $A\vec{0} = \lambda\vec{0}$ is always true for <u>any</u> λ, which is is not interesting, and not useful.]

Problem: Given a matrix A, find/compute all the eigenvalues λ & the corresponding λ-eigenvectors.

HOW?

Rephrase the definition:

λ is eigenvalue

$\iff$ system $A\vec{u} = \lambda\vec{u}$ has a nonzero solution $\vec{u} \neq \vec{0}$.

$\iff$
$$A\vec{u} - \lambda\vec{u} = \vec{0}$$
$$A\vec{u} - \lambda I\vec{u} = \vec{0}$$
$$(A - \lambda I)\vec{u} = \vec{0}$$

TRICK

has a nonzero solution $\vec{u} \neq \vec{0}$.

$\iff$ Matrix $A - \lambda I$ sends some nonzero vector $\vec{u}$ to $\vec{0}$.

$\iff$ $(A - \lambda I)^{-1}$ does not exist.

$\iff$ $\underbrace{\det(A - \lambda I) = 0}_{\text{This is useful!}}$.

Example: 2×2 matrices.

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Find all the eigenvalues of A.

From Above:

λ is an eigenvalue of

$$\Longleftrightarrow \det(A-\lambda I)=0.$$

$$\Longleftrightarrow \det\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}-\lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)=0.$$

$$\Longleftrightarrow \det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}=0.$$

$$\Longleftrightarrow (a-\lambda)(d-\lambda)-bc=0.$$

$$\Longleftrightarrow \lambda^2-(a+d)\lambda+(ad-bc)=0$$

This is called the "characteristic equation" of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Using the Qudratic Formula:

$$\lambda=\frac{(a+d)\pm\sqrt{(a+d)^2-4(ad-bc)}}{2}$$

These are the eigenvalues. But you don't need to memorize this.

Example: Eigenvalues of $\begin{pmatrix} 0.4 & 0.2 \\ 0.6 & 0.8 \end{pmatrix}$:

$$\det\left(\begin{pmatrix} 0.4 & 0.2 \\ 0.6 & 0.8 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = 0$$

$$\det\begin{pmatrix} 0.4-\lambda & 0.2 \\ 0.6 & 0.8-\lambda \end{pmatrix} = 0$$

$$(0.4-\lambda)(0.8-\lambda) - (0.2)(0.6) = 0.$$

$$\lambda^2 - 1.2\lambda + 0.32 - 0.12$$

$$\lambda^2 - 1.2\lambda + 0.2 = 0$$

$$5\lambda^2 - 6\lambda + 1 = 0$$

$$\lambda = \frac{6 \pm \sqrt{36-20}}{10}$$

$$= \frac{6 \pm 4}{10} = 1 \text{ or } 0.2 \quad \checkmark$$

Once we have the eigenvalues, we can go find the eigenvectors.

Compute the 1-eigenvectors:

$$(A - 1I)\vec{u} = \vec{0}.$$

$$\begin{pmatrix} 0.4-1 & 0.2 \\ 0.6 & 0.8-1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} -0.6 & 0.2 & 0 \\ 0.6 & -0.2 & 0 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cc|c} -0.6 & 0.2 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad ② = ② + ①$$

$$\longrightarrow \left(\begin{array}{cc|c} 1 & -1/3 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad ① = ①/(-0.6)$$

Solution of $u_1 - \frac{1}{3}u_2 = 0$:

Let $t = u_2$ be free. Then

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1/3\, t \\ t \end{pmatrix} = t\begin{pmatrix} 1/3 \\ 1 \end{pmatrix}.$$

The 1-eigenvectors form a line.
I'm going to choose my favorite vector on this line: $\vec{u} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

Check: $\begin{pmatrix} 0.4 & 0.2 \\ 0.6 & 0.8 \end{pmatrix} \underline{\begin{pmatrix} 1 \\ 3 \end{pmatrix}} = 1 \underline{\begin{pmatrix} 1 \\ 3 \end{pmatrix}}$ ✓

Next compute the (0.2)-eigenvectors:

$$(A - 0.2 I)\vec{v} = \vec{0}$$

$$\begin{pmatrix} 0.4-0.2 & 0.2 \\ 0.6 & 0.8-0.2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 0.2 & 0.2 & 0 \\ 0.6 & 0.6 & 0 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cc|c} 0.2 & 0.2 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

Solution of $v_1 + v_2 = 0$:

Let $t = v_2$ be free, then

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -t \\ t \end{pmatrix} = t \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

The (0.2)-eigenvectors form a line.
I'll choose my favorite vector
on this line: $\vec{v} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

Check: $\begin{pmatrix} 0.4 & 0.2 \\ 0.6 & 0.8 \end{pmatrix} \underline{\begin{pmatrix} -1 \\ 1 \end{pmatrix}} = 0.2 \underline{\begin{pmatrix} -1 \\ 1 \end{pmatrix}}$ ✓

We finally finished the problem
of the migrating bears.
Yes, it took a while, but now that
we know the algorithm the next
example will go much faster.