HW 6 will be due Thurs Nov 19, which is the last day of class.

There will be no Quiz 6.

Final Project (summary of what you learned) is due Tues Dec 1.

Final Topic : "Diagonalization."

Let A be square $n \times n$ matrix.

For any point $\vec{x}_0 \in \mathbb{R}^n$ we get an infinite sequence of points:

$$\vec{x}_0 \longrightarrow \vec{x}_1 = A\vec{x}_0 \longrightarrow \vec{x}_2 = A\vec{x}_1 \longrightarrow \cdots$$

Linear recurrence relation:

$$\boxed{\vec{x}_{n+1} = A\vec{x}_n}$$

Goal: Describe how this infinite sequence behaves:

- does it converge to a point?
- does it diverge to infinity?
- does it approach an asymptote?
- does it oscillate?

More precisely, we want a formula for the coordinates of the nth point.

$$\vec{x}_n = (?, ?, ?, \ldots, ?)$$

Easy Example: Let $A = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$.

Let $\vec{x}_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$. Then

$$\vec{x}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = A\vec{x}_0$$

$$= \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 3x_0 \\ 2y_0 \end{pmatrix}.$$

$$\vec{x}_2 = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 3x_0 \\ 2y_0 \end{pmatrix} = \begin{pmatrix} 9x_0 \\ 4y_0 \end{pmatrix}$$

$$\vec{x}_3 = \cdots = \begin{pmatrix} 27x_0 \\ 8y_0 \end{pmatrix}$$

$$\vdots$$

$$\vec{x}_n = \begin{pmatrix} 3^n x_0 \\ 2^n y_0 \end{pmatrix}.$$

This is an explicit formula :)

The reason this was easy:
Because the matrix $A = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$ is "diagonal," i.e., all entries off the main diagonal are zero.

Idea: Diagonal matrices are easy to work with. Suppose

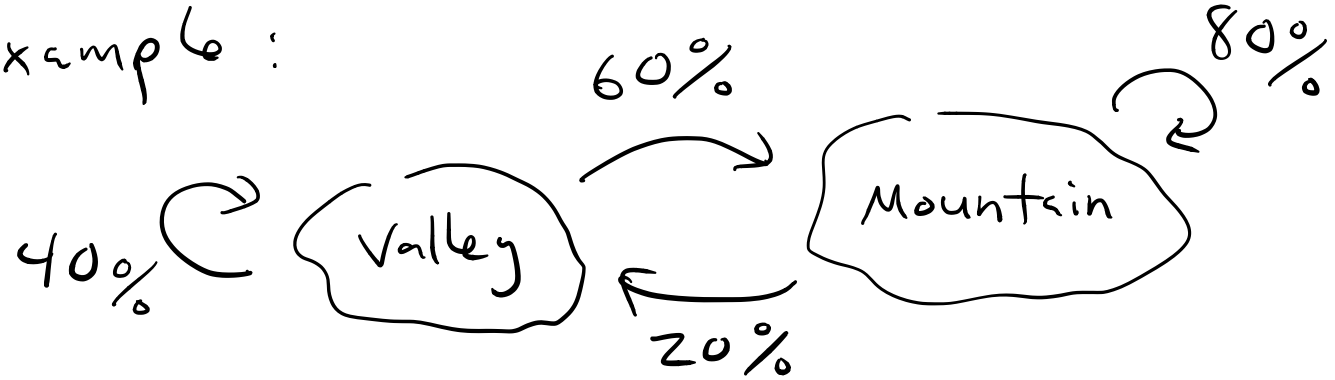
$$A = \begin{pmatrix} a_1 & & & 0 \\ & a_2 & & \\ & & \ddots & \\ 0 & & & a_m \end{pmatrix}$$

Then the powers of A are just

$$A^r = \begin{pmatrix} a_1^r & & & 0 \\ & a_2^r & & \\ & & \ddots & \\ 0 & & & a_m^r \end{pmatrix}$$

On the other hand, computing powers of a non-diagonal matrix is <u>hard</u>.

Example:



Represents year-to-year migration of of a population of bears. Let

$x_n = \#$ valley bears in year n

$y_n = \#$ mountain bears in year n

The diagram becomes a system of linear equations:

$$\begin{cases} x_{n+1} = 0.4\,x_n + 0.2\,y_n \\ y_{n+1} = 0.6\,x_n + 0.8\,y_n \end{cases}$$

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 0.4 & 0.2 \\ 0.6 & 0.8 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

$$\vec{x}_{n+1} = A\,\vec{x}_n$$

We want to "solve" this "dynamical system," i.e., we want formulas for x_n & y_n. This will depend on the "initial conditions" x_0 & y_0.

Let's just say

$$\vec{x}_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 100 \\ 0 \end{pmatrix}.$$

In the first year:

$$\vec{x}_1 = \begin{pmatrix} 0.4 & 0.2 \\ 0.6 & 0.8 \end{pmatrix} \begin{pmatrix} 100 \\ 0 \end{pmatrix} = \begin{pmatrix} 40 \\ 60 \end{pmatrix}.$$

In the second year:

$$\vec{x}_2 = \begin{pmatrix} 0.4 & 0.2 \\ 0.6 & 0.8 \end{pmatrix} \begin{pmatrix} 40 \\ 60 \end{pmatrix} = \begin{pmatrix} 28 \\ 72 \end{pmatrix}.$$

Move to the computer:

$$\vec{x}_3 = \begin{pmatrix} 25.6 \\ 74.4 \end{pmatrix}$$

$$\vec{x}_4 = \begin{pmatrix} 25.12 \\ 74.88 \end{pmatrix}$$

[You might have a guess what will happen as $n \to \infty$.]

Observe that

$$\begin{aligned}\vec{x}_n &= A\vec{x}_{n-1} \\ &= AA\vec{x}_{n-2} \\ &= AAA\vec{x}_{n-3} \\ &\vdots \\ &= \underbrace{AAA\cdots\cdots A}_{\text{how many?}}\vec{x}_0\end{aligned}$$

$$\boxed{\vec{x}_n = A^n \vec{x}_0}$$

So we are really trying to compute the powers of the matrix A.

Right now I'll just tell you

the answer & I'll explain next time why it works.

There are 3 key equations:

① $$\begin{pmatrix} 0.4 & 0.2 \\ 0.6 & 0.8 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

② $$\begin{pmatrix} 0.4 & 0.2 \\ 0.6 & 0.8 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = (0.2) \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

③ $$\begin{pmatrix} 100 \\ 0 \end{pmatrix} = 25 \begin{pmatrix} 1 \\ 3 \end{pmatrix} - 75 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

I claim that these 3 mysterious equations give us the answer:

$$\vec{x}_n = A^n \vec{x}_0$$

$$= \begin{pmatrix} 0.4 & 0.2 \\ 0.6 & 0.8 \end{pmatrix}^n \begin{pmatrix} 100 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0.4 & 0.2 \\ 0.6 & 0.8 \end{pmatrix}^n \left[25 \begin{pmatrix} 1 \\ 3 \end{pmatrix} - 75 \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right]$$

$$= 25 \begin{pmatrix} 0.4 & 0.2 \\ 0.6 & 0.8 \end{pmatrix}^n \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$- 75 \begin{pmatrix} 0.4 & 0.2 \\ 0.6 & 0.8 \end{pmatrix}^n \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

[Apply equations 1 & 2]

$$= 25 \begin{pmatrix} 1 \\ 3 \end{pmatrix} - 75 (0.2)^n \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 25 \\ 75 \end{pmatrix} + \begin{pmatrix} 75 (0.2)^n \\ -75 (0.2)^n \end{pmatrix}$$

$$= \begin{pmatrix} 25 + 75 (0.2)^n \\ 75 - 75 (0.2)^n \end{pmatrix}.$$

Summary: In year n we have

x_n = # valley bears = $25 + 75(0.2)^n$

y_n = # mtn bears = $75 - 75(0.2)^n$.

That's pretty good!

Interesting to note that

$$(0.2)^n \longrightarrow 0 \quad \text{as } n \longrightarrow \infty$$

so

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} \longrightarrow \begin{pmatrix} 25 \\ 75 \end{pmatrix} \quad \text{as } n \longrightarrow \infty.$$

This is called a "stable equilibrium." Eventually, 25% will be in the valley & 75% on the mountain.

Next time we'll discuss the details.

Jargon: The red & blue vectors above are called "eigenvectors" of the matrix $\begin{pmatrix} 0.4 & 0.2 \\ 0.6 & 0.8 \end{pmatrix}$.

"eigen" = "belongs to"

German.