**Math 210** **Fall 2020**
**Rules of Vector Arithmetic** **Drew Armstrong**

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A *vector space* $V$ is a set of "things" $\mathbf{x} \in V$, which we will call "vectors." A vector space has three algebraic operations, defined as follows:

- For any vectors $\mathbf{x}, \mathbf{y} \in V$ we have a vector called $\mathbf{x} + \mathbf{y} \in V$. Mnemonic:
$$\text{(vector)} + \text{(vector)} = \text{(vector)}$$
- For any vector $\mathbf{x} \in V$ and scalar $t \in \mathbb{R}$ we have a vector called $t\mathbf{x} \in V$. Mnemonic:
$$\text{(scalar)(vector)} = \text{(vector)}$$
- For any vectors $\mathbf{x}, \mathbf{y} \in V$ we have a scalar called $\mathbf{x} \bullet \mathbf{y} \in \mathbb{R}$. Mnemonic:
$$\text{(vector)} \bullet \text{(vector)} = \text{(scalar)}$$

The prototype is $V = \mathbb{R}^n$, i.e., the set of column vectors with $n$ real coordinates. In this case the three operations are "componentwise addition," "scalar multiplication," and "dot product," defined as follows:[1]

$$\begin{aligned} (x_1, \ldots, x_n) + (y_1, \ldots, y_n) &= (x_1 + y_1, \ldots, x_n + y_n), \\ t(x_1, \ldots, x_n) &= (tx_1, \ldots, tx_n), \\ (x_1, \ldots, x_n) \bullet (y_1, \ldots, y_n) &= x_1 y_1 + \cdots + x_n y_n. \end{aligned}$$

The three vector space operations are required to satisfy the following rules:

- There exists a vector $\mathbf{0} \in V$ with the property $\mathbf{x} + \mathbf{0} = \mathbf{0} + \mathbf{x} = \mathbf{x}$ for all $\mathbf{x} \in V$.
- We have $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$ for all $\mathbf{x}, \mathbf{y} \in V$.
- We have $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$ for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$.
- We have $0\mathbf{x} = \mathbf{0}$ and $1\mathbf{x} = \mathbf{x}$ for all $\mathbf{x} \in V$.
- We have $(s + t)\mathbf{x} = s\mathbf{x} + t\mathbf{x}$ for all $s, t \in \mathbb{R}$ and $\mathbf{x} \in V$.
- We have $t(\mathbf{x} + \mathbf{y}) = t\mathbf{x} + t\mathbf{y}$ for all $t \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in V$.
- We have $\mathbf{x} \bullet \mathbf{y} = \mathbf{y} \bullet \mathbf{x}$ for all $\mathbf{x}, \mathbf{y} \in V$.
- We have $t(\mathbf{x} \bullet \mathbf{y}) = (t\mathbf{x}) \bullet \mathbf{y} = \mathbf{x} \bullet (t\mathbf{y})$ for all $t \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in V$.
- We have $(s\mathbf{x} + t\mathbf{y}) \bullet \mathbf{z} = s(\mathbf{x} \bullet \mathbf{z}) + t(\mathbf{y} \bullet \mathbf{z})$ for all $t \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$.
- We have $\mathbf{x} \bullet \mathbf{x} \geq 0$ for all $\mathbf{x} \in V$.
- We have $\mathbf{x} \bullet \mathbf{x} = 0$ if and only if $\mathbf{x} = \mathbf{0}$.

One can check that all of these rules are satisfied for the basic operations on $\mathbb{R}^n$.

Why do we bother to define abstract vector spaces? Because there exist interesting vector spces beyond just $\mathbb{R}^n$. For example: radio signals, random variables on an experiment, states of a quantum system (in the third example we need to use complex instead of real scalars).

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[1] To save space I wrote column vectors as horizontal lists.