Math 210 **Fall 2020**
Quiz 5 **Drew Armstrong**

You have 20 minutes to write the quiz. No collaboration is allowed. When finished, you have 5 minutes to upload a pdf scan of your work to the google classroom.

Problem 1. [5 points] Consider the point $\mathbf{b} = (1,1,0)$ and the vector $\mathbf{a} = (1,2,1)$.

(a) Project[1] the point $\mathbf{b}$ onto the line $t\mathbf{a} = t(1,2,1)$.

(b) Project the point $\mathbf{b}$ onto the plane $\mathbf{a}^T\mathbf{x} = x + 2y + z = 0$. [Hint: If P is the matrix that projects onto the line then $Q = I - P$ is the matrix that projects onto the plane.]

(a): The matrix that projects onto the line $t\mathbf{a}$ is

$$P = \frac{1}{\|\mathbf{a}\|^2}\mathbf{a}\mathbf{a}^T = \frac{1}{1^2+2^2+1^2}\begin{pmatrix}1\\2\\1\end{pmatrix}\begin{pmatrix}1 & 2 & 1\end{pmatrix} = \frac{1}{6}\begin{pmatrix}1 & 2 & 1\\2 & 4 & 2\\1 & 2 & 1\end{pmatrix}.$$

The projection of the point $\mathbf{b}$ onto the line is

$$P\mathbf{b} = \frac{1}{6}\begin{pmatrix}1 & 2 & 1\\2 & 4 & 2\\1 & 2 & 1\end{pmatrix}\begin{pmatrix}1\\1\\0\end{pmatrix} = \frac{1}{6}\begin{pmatrix}3\\6\\3\end{pmatrix} = \begin{pmatrix}1/2\\1\\1/2\end{pmatrix}.$$

(a): Note that the line $t(1,2,1)$ and the plane $x + 2y + z = 0$ are orthogonal complements. Thus the matrix that projects onto the plane is

$$Q = I - P = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix} - \frac{1}{6}\begin{pmatrix}1 & 2 & 1\\2 & 4 & 2\\1 & 2 & 1\end{pmatrix} = \frac{1}{6}\begin{pmatrix}5 & -2 & -1\\-2 & 2 & -2\\-1 & -2 & 5\end{pmatrix}$$

Then the projection of the point $\mathbf{b}$ onto the plane is

$$Q\mathbf{b} = \frac{1}{6}\begin{pmatrix}5 & -2 & -1\\-2 & 2 & -2\\-1 & -2 & 5\end{pmatrix}\begin{pmatrix}1\\1\\0\end{pmatrix} = \frac{1}{6}\begin{pmatrix}3\\0\\-3\end{pmatrix} = \begin{pmatrix}1/2\\0\\-1/2\end{pmatrix}.$$

Faster solution: $Q\mathbf{b} = (I-P)\mathbf{b} = \mathbf{b} - P\mathbf{b} = (1,1,0) - (1/2,1,1/2) = (1/2,0,-1/2)$.

Slower solution: Find a matrix A whose column space is the plane. Then compute the projection matrix $Q = A(A^TA)^{-1}A^T$. Then compute $Q\mathbf{b}$.

Problem 2. [5 points] Consider the following three data points:

$$(x,y) = (0,0), (1,2), (2,1).$$

(a) Find $\hat{m}, \hat{b}$ so that $y = \hat{m}x + \hat{b}$ is the best fit line for these data points. (This should minimize the sum of the squares of the vertical errors.)

(b) Draw a picture of the data points and the best fit line.

[1] Orthogonal projection, as usual.

(a): Each data point gives a linear equation in m and b. These three equations have no solution, so we solve the normal equation to obtain least squares approximations $\hat{m}$ and $\hat{b}$:

$$\begin{aligned}
\begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} &= \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \\
\begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} \hat{m} \\ \hat{b} \end{pmatrix} &= \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \\
\begin{pmatrix} 5 & 3 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} \hat{m} \\ \hat{b} \end{pmatrix} &= \begin{pmatrix} 4 \\ 3 \end{pmatrix} \\
\begin{pmatrix} \hat{m} \\ \hat{b} \end{pmatrix} &= \begin{pmatrix} 5 & 3 \\ 3 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 4 \\ 3 \end{pmatrix} \\
&= \frac{1}{6} \begin{pmatrix} 3 & -3 \\ -3 & 5 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix} \\
&= \frac{1}{6} \begin{pmatrix} 3 \\ 3 \end{pmatrix} \\
&= \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}.
\end{aligned}$$

The best fit line is $y = \hat{m}x + \hat{b} = x/2 + 1/2$.

(b): Here is a picture: