Math 210 **Fall 2020**
Quiz 4 **Drew Armstrong**

You have 20 minutes to write the quiz. No collaboration is allowed. When finished, you have 5 minutes to upload a pdf scan of your work to the google classroom.

Problem 1. [4 points]

Use Gaussian elimination to compute the inverse of the following matrix:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}.$$

Show your work for full credit.

We form the augmented matrix $(A|I)$ and then use Gaussian elimination to obtain $(I|A^{-1})$:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 2 & 1 & -3 & 0 & 1 \end{array}\right) \quad \begin{array}{l} (2) = (2) - 2(1) \\ (3) = (3) - 3(1) \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right) \quad (3) = (3) - 2(2)$$

We conclude that

$$A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{pmatrix}.$$

Problem 2. [6 points]

(a) Find some nonzero 2×2 matrix A such that A^{-1} does not exist.

(b) Find some nonzero 2×2 matrices A and B such that $AB = BA$.

(c) Find a matrix A such that $A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $A\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$.

(a): Recall that the inverse of a general 2×2 matrix is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

and if $ad - bc = 0$ then the inverse does not exist.[1] Thus we only have to find some numbers a, b, c, d that are not all zero and satisfy $ad - bc = 0$. For example:

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

(b): In general we could let A have entries a, b, c, d and let B have entries e, f, g, h. Then express the equation $AB = BA$ as a system of four linear equations in the eight unknowns, which has many many solutions. But there are easier tricks. For example: Let $B = I$, so that

$$AI = A = IA.$$

Another Example: Let $B = A$, so that

$$AB = A^2 = BA.$$

(c): This time there is a unique answer. Note that the matrix A must have shape 3×2. Furthermore it must have first column[2]

$$A \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

and second column

$$A \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix},$$

so the unique answer is

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

[1]We note that $ad - bc = 0$ if and only if the two rows are parallel; equivalently, if and only if the two columns are parallel. [Special Case: The zero vector is parallel to every vector.] So you only need to find two parallel vectors in $\mathbb{R}^2$, at least one of which is not the zero vector.

[2]Recall that if $f : \mathbb{R}^n \to \mathbb{R}^m$ is any linear function then the jth column of the $m \times n$ matrix $[f]$ is the vector $f(\mathbf{e}_j) \in \mathbb{R}^m$, where $\mathbf{e}_j$ is the jth standard basis vector of $\mathbb{R}^n$.