Math 210 **Fall 2020**
Homework 6 **Drew Armstrong**

Problem 1. An Important Formula. Let $\mathbf{u}, \mathbf{v} \neq \mathbf{0}$ be eigenvectors of a square matrix A:

$$A\mathbf{u} = \lambda\mathbf{u} \quad \text{ and } A\mathbf{v} = \mu\mathbf{v}.$$

(a) Show that $A^n\mathbf{u} = \lambda^n\mathbf{u}$ for all integers $n \geq 1$.
(b) Use part (a) to show that $A^n(a\mathbf{u} + b\mathbf{v}) = s\lambda^n\mathbf{u} + b\mu^n\mathbf{v}$ for all scalars a, b.

(a): We will use a method called *proof by induction*. First note that the statement is true for $n = 1$ because $A^1 = A$ and $\lambda^1 = \lambda$ by definition. Now suppose that we have $A^n\mathbf{u} = \lambda^n\mathbf{u}$ for some $n \geq 1$. Then it follows that

$$\begin{aligned} A^{n+1}\mathbf{u} &= (AA^n)\mathbf{u} \\ &= A(A^n\mathbf{u}) \\ &= A(\lambda^n\mathbf{u}) \\ &= \lambda^n(A\mathbf{u}) \\ &= \lambda^n(\lambda\mathbf{u}) \\ &= \lambda^{n+1}\mathbf{u}. \end{aligned}$$

(b): This follows immediately from part (a):

$$A^n(a\mathbf{u} + b\mathbf{v}) = a(A^n\mathbf{u}) + b(A^n\mathbf{v}) = a(\lambda^n\mathbf{u}) + b(\mu^n\mathbf{v}).$$

Problem 2. A Projection and a Reflection. Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$ be vectors in the plane satisfying $\mathbf{a}^T\mathbf{b} = 0$, and let P be the matrix that projects onto the line $t\mathbf{a}$:

$$P = \frac{1}{\|\mathbf{a}\|^2}\mathbf{a}\mathbf{a}^T.$$

(a) Show that $\mathbf{a}$ and $\mathbf{b}$ are eigenvectors of P. What are the corresponding eigenvalues?
(b) Show that $\mathbf{a}$ and $\mathbf{b}$ are eigenvectors of $F = 2P - I$. What are the eigenvalues?
(c) Describe what the matrix F does geometrically.

(a): First we compute $P\mathbf{a}$, using the fact that $\mathbf{a}^T\mathbf{a} = \|\mathbf{a}\|^2$:

$$P\mathbf{a} = \frac{1}{\|\mathbf{a}\|^2}(\mathbf{a}\mathbf{a}^T)\mathbf{a} = \frac{1}{\|\mathbf{a}\|^2}\mathbf{a}(\mathbf{a}^T\mathbf{a}) = \frac{1}{\cancel{\|\mathbf{a}\|^2}}\mathbf{a}\cancel{\|\mathbf{a}\|^2} = \mathbf{a} = 1\mathbf{a}.$$

Thus $\mathbf{a}$ is a 1-eigenvector of P. Next we compute $P\mathbf{b}$, using the fact that $\mathbf{a}^T\mathbf{b} = 0$:

$$P\mathbf{b} = \frac{1}{\|\mathbf{a}\|^2}(\mathbf{a}\mathbf{a}^T)\mathbf{b} = \frac{1}{\|\mathbf{a}\|^2}\mathbf{a}(\mathbf{a}^T\mathbf{b}) = \frac{1}{\|\mathbf{a}\|^2}\mathbf{a}0 = \mathbf{0} = 0\mathbf{b}.$$

Thus $\mathbf{b}$ is a 0-eigenvector of P. See the course notes for discussion.

(b): From part (a) we have

$$\begin{aligned} F\mathbf{a} &= (2P - I)\mathbf{a} = 2P\mathbf{a} - \mathbf{a} = 2\mathbf{a} - \mathbf{a} = 1\mathbf{a}, \\ F\mathbf{b} &= (2P - I)\mathbf{b} = 2P\mathbf{b} - \mathbf{b} = 2\mathbf{0} - \mathbf{b} = -1\mathbf{b}. \end{aligned}$$

We conclude that $\mathbf{a}$ is a 1-eigenvector and $\mathbf{b}$ is a (-1)-eigenvector of F. [More generally, for any polynomial expression $f(x)$ we have $f(P)\mathbf{a} = f(1)\mathbf{a}$ and $f(P)\mathbf{b} = f(0)\mathbf{b}$. In this example we had $f(x) = 2x - 1$.]

(c): F is the (orthogonal) reflection across the line $t\mathbf{a}$. See the course notes for discussion.

Problem 3. Eigenvalues of a Rotation. Consider again the rotation matrix:

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

(a) Use the characteristic equation to find the complex eigenvalues of R_θ.
(b) For which values of θ are the eigenvalues real? Find the eigenvectors in each case.

(a): We will write $c = \cos\theta$ and $s = \sin\theta$, so that $c^2 + s^2 = 1$. The characteristic equation is

$$\begin{aligned} \det(R_\theta - \lambda I) &= 0 \\ \det\begin{pmatrix} c-\lambda & -s \\ s & c-\lambda \end{pmatrix} &= 0 \\ (c-\lambda)(c-\lambda) - (-s)s &= 0 \\ \lambda^2 - 2c\lambda + c^2 + s^2 &= 0 \\ \lambda^2 - 2c\lambda + 1 &= 0. \end{aligned}$$

Then we use the quadratic formula to obtain the eigenvalues:

$$\begin{aligned} \lambda &= (2c \pm \sqrt{(-2c)^2 - 4})/2 \\ &= c \pm \sqrt{c^2 - 1} \\ &= c \pm \sqrt{-s^2} \\ &= c \pm s\sqrt{-1} \\ &= \cos\theta \pm i\sin\theta. \end{aligned}$$

(b): A complex number $a + ib$ is real if and only if $b = 0$. The eigenvalues $\lambda = \cos\theta \pm i\sin\theta$ are real if and only if $\sin\theta = 0$, i.e., if and only if $\theta = 0$ or $\theta = 180°$. In the first case we have

$$R_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I,$$

so that **every nonzero vector** in $\mathbb{R}^2$ is a 1-eigenvector. In the second case we have

$$R_{180°} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I,$$

so that **every nonzero vector** in $\mathbb{R}^2$ is a (-1)-eigenvector. For any other value of θ the rotation R_θ has **no real eigenvectors**. See the course notes for discussion.

Problem 4. Diagonalizing a Matrix. Consider the following 2×2 matrix:

$$A = \frac{1}{6}\begin{pmatrix} 5 & 4 \\ 2 & -2 \end{pmatrix}.$$

(a) Solve the characteristic equation to find the eigenvalues λ, μ.
(b) Solve the equations $(A - \lambda I)\mathbf{u} = \mathbf{0}$ and $(A - \mu I)\mathbf{v} = \mathbf{0}$ to find eigenvectors $\mathbf{u}, \mathbf{v}$.
(c) Draw a picture of the eigenspaces in the plane.

(a): The characteristic equation is

$$\begin{aligned} \det(A - \lambda I) &= 0 \\ \det \begin{pmatrix} 5/6 - \lambda & 4/6 \\ 2/6 & -2/6 - \lambda \end{pmatrix} &= 0 \\ (5/6 - \lambda)(-2/6 - \lambda) - (4/6)(2/6) &= 0 \\ \lambda^2 - (3/6)\lambda - 10/36 - 8/36 &= 0 \\ \lambda^2 - (1/2)\lambda - 1/2 &= 0. \end{aligned}$$

Then we use the quadratic formula to obtain the eigenvalues:

$$\begin{aligned} \lambda &= (1/2 \pm \sqrt{(-1/2)^2 - 4(-1/2)})/2 \\ &= (1/2 \pm \sqrt{9/4})/2 \\ &= (1/2 \pm 3/2)/2 \\ &= 1 \text{ or } -1/2. \end{aligned}$$

(b): The eigenspace E_1 is the solution of the linear system $(A - 1I)\mathbf{u} = \mathbf{0}$:

$$(A - 1I|\mathbf{0}) = \left(\begin{array}{cc|c} 5/6 - 1 & 4/6 & 0 \\ 2/6 & -2/6 - 1 & 0 \end{array}\right) \xrightarrow{\text{RREF}} \left(\begin{array}{cc|c} 1 & -4 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

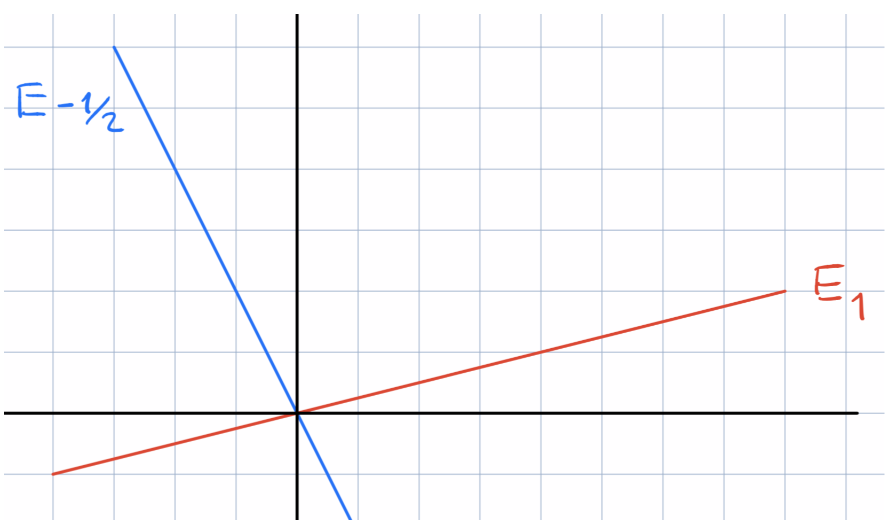
We conclude that E_1 is the line $\mathbf{u} = t(4, 1)$.

The eigenspace $E_{-1/2}$ is the solution of the linear system $(A + (1/2)I)\mathbf{u} = \mathbf{0}$:

$$(A + (1/2)I|\mathbf{0}) = \left(\begin{array}{cc|c} 5/6 + 1/2 & 4/6 & 0 \\ 2/6 & -2/6 + 1/2 & 0 \end{array}\right) \xrightarrow{\text{RREF}} \left(\begin{array}{cc|c} 1 & 1/2 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

We conclude that $E_{-1/2}$ is the line $\mathbf{v} = t(1, -2)$.

(c): Picture:



Problem 5. Two Dynamical Systems. Let A be the same matrix from Problem 4.

(a) Express the vector $(2, 5)$ as $a\mathbf{u} + b\mathbf{v}$ where $\mathbf{u}, \mathbf{v}$ are the eigenvectors of A.

(b) **A Discrete Dynamical System.** Let the points $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots$ in $\mathbb{R}^2$ be defined by

$$\mathbf{x}_0 = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_{n+1} = A\mathbf{x}_n.$$

Use part (a) and Problem 4 to find an explicit formula for $\mathbf{x}_n$. [Recall that the general solution looks like $\mathbf{x}_n = a\lambda^n\mathbf{u} + b\mu^n\mathbf{v}$.]

(c) **A Continuous Dynamical System.** Let the path $\mathbf{x}(t)$ in $\mathbb{R}^2$ be defined by[1]

$$\mathbf{x}(0) = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \quad \text{and} \quad \mathbf{x}'(t) = A\mathbf{x}(t).$$

Use part (a) and Problem 4 to find an explicit formula for $\mathbf{x}(t)$. [Recall that the general solution looks like $\mathbf{x}(t) = ae^{\lambda t}\mathbf{u} + be^{\mu t}\mathbf{v}$.]

(a): We will use $\mathbf{u} = (4,1)$ and $\mathbf{v} = (1,-2)$, so that

$$\begin{aligned}
a\begin{pmatrix} 4 \\ 1 \end{pmatrix} + b\begin{pmatrix} 1 \\ -2 \end{pmatrix} &= \begin{pmatrix} 2 \\ 5 \end{pmatrix} \\
\begin{pmatrix} 4 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} &= \begin{pmatrix} 2 \\ 5 \end{pmatrix} \\
\begin{pmatrix} a \\ b \end{pmatrix} &= \begin{pmatrix} 4 & 1 \\ 1 & -2 \end{pmatrix}^{-1}\begin{pmatrix} 2 \\ 5 \end{pmatrix} \\
&= -\frac{1}{9}\begin{pmatrix} -2 & -1 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} 2 \\ 5 \end{pmatrix} \\
&= -\frac{1}{9}\begin{pmatrix} -9 \\ 18 \end{pmatrix} \\
&= \begin{pmatrix} 1 \\ -2 \end{pmatrix}
\end{aligned}$$

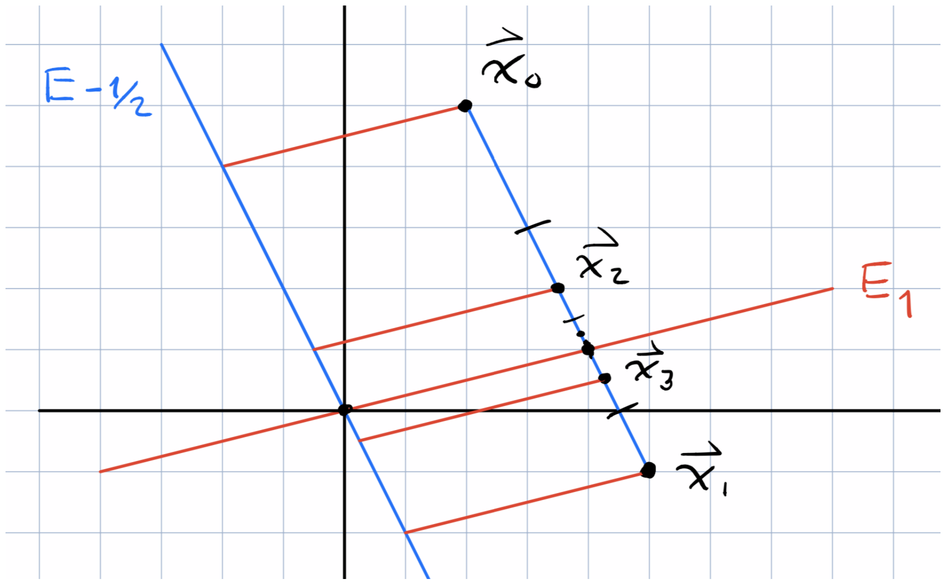
We conclude that $(2,5) = 1(4,1) - 2(1,-2) = (4,1) + (-2,4)$ where $(4,1)$ is a 1-eigenvector and $(-2,4)$ is a $(-1/2)$-eigenvector of A.

(b): The solution of $\mathbf{x}_0 = (2,5)$ and $\mathbf{x}_{n+1} = A\mathbf{x}_n$ is

$$\begin{aligned}
\mathbf{x}_n &= A^n\mathbf{x}_0 \\
&= A^n\left(\begin{pmatrix} 4 \\ 1 \end{pmatrix} + \begin{pmatrix} -2 \\ 4 \end{pmatrix}\right) \\
&= A^n\begin{pmatrix} 4 \\ 1 \end{pmatrix} + A^n\begin{pmatrix} -2 \\ 4 \end{pmatrix} \\
&= 1^n\begin{pmatrix} 4 \\ 1 \end{pmatrix} + (-1/2)^n\begin{pmatrix} -2 \\ 4 \end{pmatrix} \\
&= \begin{pmatrix} 4 - 2(-1/2)^n \\ 1 + 4(-1/2)^n \end{pmatrix}.
\end{aligned}$$

Here is a picture:

[1] If the position $\mathbf{x}(t)$ has coordinates $x(t)$ and $y(t)$ then the velocity $\mathbf{x}'(t)$ has coordinates $x'(t)$ and $y'(t)$.



(c): The solution of $\mathbf{x}(0) = (2, 5)$ and $\mathbf{x}'(t) = A\mathbf{x}(t)$ is

$$\begin{aligned}
\mathbf{x}(t) &= e^{At}\mathbf{x}(0) \\
&= e^{At}\left(\begin{pmatrix} 4 \\ 1 \end{pmatrix} + \begin{pmatrix} -2 \\ 4 \end{pmatrix}\right) \\
&= e^{At}\begin{pmatrix} 4 \\ 1 \end{pmatrix} + e^{At}\begin{pmatrix} -2 \\ 4 \end{pmatrix} \\
&= e^{1t}\begin{pmatrix} 4 \\ 1 \end{pmatrix} + e^{(-1/2)t}\begin{pmatrix} -2 \\ 4 \end{pmatrix} \\
&= \begin{pmatrix} 4e^{t} - 2e^{-t/2} \\ e^{t} + 4e^{-t/2} \end{pmatrix}.
\end{aligned}$$

Here is a picture: