**Math 210** **Fall 2020**
**Homework 4** **Drew Armstrong**

---

**Problem 1. Matrices are Linear Functions.** An $m \times n$ matrix $A$ can be viewed as a function from $\mathbb{R}^n$ to $\mathbb{R}^m$, that sends each vector $\mathbf{x} \in \mathbb{R}^n$ to the vector $A\mathbf{x} \in \mathbb{R}^m$. Show that this function satisfies the following property:

$$A(s\mathbf{u} + t\mathbf{v}) = sA\mathbf{u} + tA\mathbf{v} \quad \text{for all } s, t \in \mathbb{R} \text{ and } \mathbf{u}, \mathbf{v} \in \mathbb{R}^n.$$

[Hint: Let $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \mathbb{R}^m$ be the column vectors of $A$. Then by definition we have $A\mathbf{x} = x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n$ for any vector $\mathbf{x} = (x_1, \ldots, x_n)$.]

*Proof.* Let $\mathbf{u} = (u_1, \ldots, u_n)$ and $\mathbf{v} = (v_1, \ldots, v_n)$ and let $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \mathbb{R}^m$ be the column vectors of $A$. Then **by definition** we have

$$\begin{aligned} A\mathbf{u} &= u_1\mathbf{a}_1 + u_2\mathbf{a}_2 + \cdots + u_n\mathbf{a}_n, \\ A\mathbf{v} &= v_1\mathbf{a}_1 + v_2\mathbf{a}_2 + \cdots + v_n\mathbf{a}_n, \end{aligned}$$

and hence for any $s, t \in \mathbb{R}$ we have

$$\begin{aligned} sA\mathbf{u} + tA\mathbf{v} &= s(u_1\mathbf{a}_1 + u_2\mathbf{a}_2 + \cdots + u_n\mathbf{a}_n) + t(v_1\mathbf{a}_1 + v_2\mathbf{a}_2 + \cdots + v_n\mathbf{a}_n) \\ &= (su_1 + tv_1)\mathbf{a}_1 + (su_2 + tv_2)\mathbf{a}_2 + \cdots + (su_n + tv_n)\mathbf{a}_n. \end{aligned}$$

On the other hand, **by definition** we have $s\mathbf{u} + t\mathbf{v} = (su_1 + tv_1, \ldots, su_n + tv_n)$, and hence

$$A(s\mathbf{u} + t\mathbf{v}) = (su_1 + tv_1)\mathbf{a}_1 + (su_2 + tv_2)\mathbf{a}_2 + \cdots + (su_n + tv_n)\mathbf{a}_n.$$

□

[Remark: This is a boring computation, but someone had to do it. The fact that $\mathbf{x} \mapsto A\mathbf{x}$ is a linear function is important because we use this property to **define** matrix multiplication.]

**Problem 2. Matching Shapes.** Let $A$ be a $3 \times 2$ matrix, let $B$ be a $3 \times 3$ matrix, let $\mathbf{x}$ be a $2 \times 1$ matrix, and let $\mathbf{y}$ be a $3 \times 1$ matrix. All of the entries of these matrices are equal to 1. Compute the following matrices or say why they don't exist:

$$AB, \quad BA, \quad A^TB, \quad \mathbf{x}^T\mathbf{y}, \quad \mathbf{x}^T\mathbf{x}, \quad \mathbf{x}\mathbf{x}^T, \quad \mathbf{y}^TA\mathbf{x}, \quad \mathbf{x}^TA^TB\mathbf{y}.$$

Explicitly, we have

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

We note that $AB$ and $\mathbf{x}^T\mathbf{y}$ do not exist because

$$\begin{aligned} (\#\text{ cols } A) &= 2 \neq 3 = (\#\text{ rows } B), \\ (\#\text{ cols } \mathbf{x}^T) &= 2 \neq 3 = (\#\text{ rows } \mathbf{y}). \end{aligned}$$

Here are the rest of the computations:

$$BA = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1+1+1 & 1+1+1 \\ 1+1+1 & 1+1+1 \\ 1+1+1 & 1+1+1 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 3 & 3 \\ 3 & 3 \end{pmatrix},$$

$$A^TB = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1+1+1 & 1+1+1 & 1+1+1 \\ 1+1+1 & 1+1+1 & 1+1+1 \end{pmatrix} = \begin{pmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \end{pmatrix},$$

$$\mathbf{x}^T\mathbf{x} = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1 + 1 = 2,$$

$$\mathbf{x}\mathbf{x}^T = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

$$\mathbf{y}^TA\mathbf{x} = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} = 2 + 2 + 2 = 6,$$

$$\mathbf{x}^TA^TB\mathbf{y} = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 9 \\ 9 \end{pmatrix} = 9 + 9 = 18.$$

**Problem 3. Special Matrices.** Find specific $2 \times 2$ matrices with the following properties:

(a) $N \neq 0$ and $N^2 = 0$,
(b) $F \neq I$ and $F^2 = I$,
(c) $P \neq 0$ and $P \neq I$ and $P^2 = P$,
(d) $R \neq I$ and $R^2 \neq I$ and $R^3 \neq I$ and $R^4 = I$.

[Remark: Each part has infinitely many correct answers. In fact, if $B$ is a solution to one of these problems then $ABA^{-1}$ is also a solution for any invertible $A$. Changing $B$ to $ABA^{-1}$ is called a *change of coordinates*. It doesn't really affect what the matrix does.]

(a): Any matrix of the form $N = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$ will work.

(b): Any reflection matrix will work. For example: $F = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

(c): Any projection matrix will work. For example: $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

(d): Rotation by $90°$ will work: $R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

**Problem 4. Computing a Matrix Inverse.** Consider the following matrix:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 3 & 4 \end{pmatrix}.$$

(a) Compute the RREF of the matrix $(A|I)$, which has the form $(I|B)$ for some $B$.
(b) Check that $AB = I$ and $BA = I$.

(c) Use the matrix $B$ to solve the following linear system, without doing any extra work:

$$\begin{cases} x + y + z = 3, \\ x + 2y + 2z = 5, \\ x + 3y + 4z = 4. \end{cases}$$

[Hint: Write the system as $A\mathbf{x} = \mathbf{b}$. Multiply on the left by $B$.]

(a): Here is the computation:

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 & 1 & 0 \\ 1 & 3 & 4 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 2 & 3 & -1 & 0 & 1 \end{array}\right) \quad \begin{array}{l} (2) = (2) - (1) \\ (3) = (3) - (1) \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right) \quad (3) = (3) - 2(2)$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & -2 & 3 & -1 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right) \quad \begin{array}{l} (1) = (1) - (3) \\ (2) = (2) - (3) \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 1 & 0 & -2 & 3 & -1 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right) \quad (1) = (1) - (2)$$

(b): I have checked it.

(c): First we convert the system to a matrix equation $A\mathbf{x} = \mathbf{b}$. Then we multiply both sides on the left by the inverse matrix $B = A^{-1}$:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 & 0 \\ -2 & 3 & -1 \\ 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 \\ -2 & 3 & -1 \\ 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 \\ -2 & 3 & -1 \\ 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ -3 \end{pmatrix}.$$

**Problem 5. Invertibility of Matrices.** Prove the following statements:

(a) If $A^{-1}$ exists then $A\mathbf{x} = A\mathbf{y}$ implies $\mathbf{x} = \mathbf{y}$.

(b) If $A\mathbf{x} = \mathbf{0}$ for some $\mathbf{x} \neq \mathbf{0}$ then $A^{-1}$ does not exist. [Hint: Use part (a) and the fact that $A\mathbf{0} = \mathbf{0}$ for any matrix $A$.]

(c) If $A^{-1}$ exists then $(A^T)^{-1}$ exists. [Hint: It is a general fact that $(AB)^T = B^T A^T$ for any matrices $A, B$. Substitute $B = A^{-1}$ into this formula.]

(d) If $A$ and $B$ are square of the same size, and if $A^{-1}$ and $B^{-1}$ both exist, then $(AB)^{-1}$ exists. [Hint: Show that the matrix $B^{-1}A^{-1}$, which exists, is the desired inverse.]

(a): If $A\mathbf{x} = A\mathbf{y}$ and if $A^{-1}$ exists then we can multiply on the left to obtain

$$\begin{aligned} A\mathbf{x} &= A\mathbf{y} \\ A^{-1}A\mathbf{x} &= A^{-1}A\mathbf{y} \\ I\mathbf{x} &= I\mathbf{y} \\ \mathbf{x} &= \mathbf{y}. \end{aligned}$$

(b): Let $A\mathbf{x} = \mathbf{0}$ for some $\mathbf{x} \neq \mathbf{0}$ and assume for contradiction that $A^{-1}$ exists. Then by multiplying on the left by $A^{-1}$ we obtain a contradiction

$$\begin{aligned} A\mathbf{x} &= \mathbf{0} \\ A^{-1}A\mathbf{x} &= A^{-1}\mathbf{0} \\ I\mathbf{x} &= \mathbf{0} \\ \mathbf{x} &= \mathbf{0}. \end{aligned}$$

[Remark: More generally, if $f$ is any kind of function that sends two different points to the same place, say $f(x) = f(y)$ for some $x \neq y$, then this function cannot have an inverse.]

(c): Suppose that $A^{-1}$ exists. If $B$ is some matrix such that $AB$ is defined then we always have the identity $(AB)^T = B^T A^T$. Now substitute $B = A^{-1}$ to obtain

$$\begin{aligned} (AB)^T &= B^T A^T \\ (AA^{-1})^T &= (A^{-1})^T A^T \\ I^T &= (A^{-1})^T A^T \\ I &= (A^{-1})^T A^T. \end{aligned}$$

In other words, $A^T$ is invertible with inverse $(A^{-1})^T$.

(d): If $A^{-1}$, $B^{-1}$ and $AB$ exist then we have

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I.$$

In other words, $AB$ is invertible with inverse $B^{-1}A^{-1}$. [Intuition: The function $AB$ "does $B$ first, then does $A$." The function $B^{-1}A^{-1}$ "undoes $A$ first, then undoes $B$."]

[Remark: It is important to remember that matrix multiplication is not commutative. For example, the following proof is **wrong**:

$$(AB)(B^{-1}A^{-1}) = (AA^{-1})(BB^{-1}) = II = I.$$

We are not allowed to move the rightmost $A^{-1}$ past the $B$'s.]