Math 210 **Fall 2020**
Homework 4 **Drew Armstrong**

Problem 1. Matrices are Linear Functions. An $m \times n$ matrix A can be viewed as a function from $\mathbb{R}^n$ to $\mathbb{R}^m$, that sends each vector $\mathbf{x} \in \mathbb{R}^n$ to the vector $A\mathbf{x} \in \mathbb{R}^m$. Show that this function satisfies the following property:

$$A(s\mathbf{u} + t\mathbf{v}) = sA\mathbf{u} + tA\mathbf{v} \quad \text{for all } s, t \in \mathbb{R} \text{ and } \mathbf{u}, \mathbf{v} \in \mathbb{R}^n.$$

[Hint: Let $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \mathbb{R}^m$ be the column vectors of A. Then by definition we have $A\mathbf{x} = x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n$ for any vector $\mathbf{x} = (x_1, \ldots, x_n)$.]

Problem 2. Matching Shapes. Let A be a 3×2 matrix, let B be a 3×3 matrix, let $\mathbf{x}$ be a 2×1 matrix, and let $\mathbf{y}$ be a 3×1 matrix. All of the entries of these matrices are equal to 1. Compute the following matrices or say why they don't exist:

$$AB, \quad BA, \quad A^TB, \quad \mathbf{x}^T\mathbf{y}, \quad \mathbf{x}^T\mathbf{x}, \quad \mathbf{x}\mathbf{x}^T, \quad \mathbf{y}^TA\mathbf{x}, \quad \mathbf{x}^TA^TB\mathbf{y}.$$

Problem 3. Special Matrices. Find specific 2×2 matrices with the following properties:

(a) $N \neq 0$ and $N^2 = 0$,
(b) $F \neq I$ and $F^2 = I$,
(c) $P \neq 0$ and $P \neq I$ and $P^2 = P$,
(d) $R \neq I$ and $R^2 \neq I$ and $R^3 \neq I$ and $R^4 = I$.

Problem 4. Computing a Matrix Inverse. Consider the following matrix:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 3 & 4 \end{pmatrix}.$$

(a) Compute the RREF of the matrix $(A|I)$, which has the form $(I|B)$ for some B.
(b) Check that $AB = I$ and $BA = I$.
(c) Use the matrix B to solve the following linear system, without doing any extra work:

$$\begin{cases} x + y + z = 3, \\ x + 2y + 2z = 5, \\ x + 3y + 4z = 4. \end{cases}$$

[Hint: Write the system as $A\mathbf{x} = \mathbf{b}$. Multiply on the left by B.]

Problem 5. Invertibility of Matrices. Prove the following statements:

(a) If A^{-1} exists then $A\mathbf{x} = A\mathbf{y}$ implies $\mathbf{x} = \mathbf{y}$.
(b) If $A\mathbf{x} = \mathbf{0}$ for some $\mathbf{x} \neq \mathbf{0}$ then A^{-1} does not exist. [Hint: Use part (a) and the fact that $A\mathbf{0} = \mathbf{0}$ for any matrix A.]
(c) If A^{-1} exists then $(A^T)^{-1}$ exists. [Hint: It is a general fact that $(AB)^T = B^TA^T$ for any matrices A, B. Substitute $B = A^{-1}$ into this formula.]
(d) If A and B are square of the same size, and if A^{-1} and B^{-1} both exist, then $(AB)^{-1}$ exists. [Hint: Show that the matrix $B^{-1}A^{-1}$, which exists, is the desired inverse.]