Math 210 **Fall 2020**

Homework 3 **Drew Armstrong**

Problem 1. Gaussian Elimination. Solve the following system by converting it to a matrix and then putting the matrix in Reduced Row Echelon Form:

$$\begin{array}{c} (1) \\ (2) \\ (3) \end{array} \begin{cases} x + 2y + 3z = 4, \\ x + 2y + 4z = 6, \\ x + 2y + 5z = 8. \end{cases}$$

Does the solution have the expected number of dimensions? Why or why not?

Solution. Before doing anything, we expect that 3 linear equations in 3 unknowns will have a $3-3=0$ dimensional solution, i.e., the solution will be a point. Now we write the system as a matrix and perform Gaussian elimination:

$$\begin{array}{cl} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 6 \\ 1 & 2 & 5 & 8 \end{pmatrix} & \begin{array}{l} (1) \\ (2) \\ (3) \end{array} \end{array}$$

$$\begin{array}{cl} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{pmatrix} & \begin{array}{l} (1) \\ (2) = (2) - 1(1) \\ (3) = (3) - 1(1) \end{array} \end{array}$$

$$\begin{array}{cl} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{array}{l} (1) \\ (2) \\ (3) = (3) - 1(2) \end{array} \end{array}$$

$$\begin{array}{cl} \begin{pmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{array}{l} (1) = (1) - 3(2) \\ (2) \\ (3) \end{array} \end{array}$$

Then we convert the RREF of the matrix back into a system of linear equations:

$$\begin{cases} x + 2y + 0 = -2, \\ 0 + 0 + z = 2, \\ 0 + 0 + 0 = 0. \end{cases}$$

We observe that x, z are pivot variables and y is free. To clean up the notation we define $t = z$. Then the solution is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2-2t \\ t \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix} + t \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}.$$

This is a 1-plane (line) in 3-dimensional space, which is not what we expected. The reason this happened is because the three original equations had a **nontrivial linear relation**, which caused a row of zeroes in the RREF. In the following bonus discussion we will find this relation.

Bonus Discussion. Row relations in a matrix A correspond to column relations in the transposed matrix A^T. To find all column relations in A^T we compute the RREF:

$$\begin{pmatrix}1&1&1\\2&2&2\\3&4&5\\4&6&8\end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix}1&0&-1\\0&1&2\\0&0&0\\0&0&0\end{pmatrix}$$

In RREF(A^T) we observe that

$$-1(\text{column } 1) + 2(\text{column } 2) = (\text{column } 3).$$

Since **column relations are unchanged by row operations**, the same relation holds in the matrix A^T. Therefore in the original matrix A we have

$$-1(\text{row } 1) + 2(\text{row } 2) = (\text{row } 3).$$

See Problem 3 for another example like this.

Problem 2. More Gaussian Elimination. Solve the following system by converting it to a matrix and then putting the matrix in Reduced Row Echelon Form:

$$\begin{array}{c} (1) \\ (2) \\ (3) \end{array} \left\{ \begin{array}{ccccccccccc} x_1 & + & 2x_2 & + & x_3 & + & 0 & + & 2x_5 & = & 1, \\ x_1 & + & 2x_2 & + & 2x_3 & + & -3x_4 & + & 3x_5 & = & 1, \\ x_1 & + & 2x_2 & + & 0 & + & 3x_4 & + & 2x_5 & = & 3. \end{array} \right.$$

Does the solution have the expected number of dimensions? Why or why not?

Solution. Before doing anything, we expect that 3 linear equations in 5 unknowns will have a $5 - 3 = 2$ dimensional solution, i.e., the solution will be a 2-plane living in 5-dimensional space. Now we write the system as a matrix and perform Gaussian elimination:

$$\begin{pmatrix}1&2&1&0&2&1\\1&2&2&-3&3&1\\1&2&0&3&2&3\end{pmatrix}\begin{array}{l}(1)\\(2)\\(3)\end{array}$$

$$\begin{pmatrix}1&2&1&0&2&1\\0&0&1&-3&1&0\\0&0&-1&3&0&2\end{pmatrix}\begin{array}{l}(1)\\(2)=(2)-1(1)\\(3)=(3)-1(1)\end{array}$$

$$\begin{pmatrix}1&2&1&0&2&1\\0&0&1&-3&1&0\\0&0&0&0&1&2\end{pmatrix}\begin{array}{l}(1)\\(2)\\(3)=(3)+1(2)\end{array}$$

$$\begin{pmatrix}1&2&1&0&0&-3\\0&0&1&-3&0&-2\\0&0&0&0&1&2\end{pmatrix}\begin{array}{l}(1)=(1)-2(3)\\(2)=(2)-1(3)\\(3)\end{array}$$

$$\begin{pmatrix}1&2&0&3&0&-1\\0&0&1&-3&0&-2\\0&0&0&0&1&2\end{pmatrix}\begin{array}{l}(1)=(1)-1(2)\\(2)\\(3)\end{array}$$

Then we convert the RREF of the matrix back into a system of linear equations:

$$\begin{cases} x_1 &+& 2x_2 &+& 0 &+& 3x_4 &+& 0 &=& -1, \\ 0 &+& 0 &+& x_3 &+& -3x_4 &+& 0 &=& -2, \\ 0 &+& 0 &+& 0 &+& 0 &+& x_5 &=& 2. \end{cases}$$

We observe that x_1, x_3, x_5 are pivot variables and x_2, x_4 are free. To clean up the notation we define $s = x_2$ and $t = x_4$. Then the solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -1-2s-3t \\ s \\ -2+3t \\ t \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ -2 \\ 0 \\ 2 \end{pmatrix} + s \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -3 \\ 0 \\ 3 \\ 1 \\ 0 \end{pmatrix}.$$

This is a 2-plane in 5-dimensional space, as expected.

Problem 3. Column Relations. Put the following matrix in Reduced Row Echelon Form:

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 3 & 4 & 6 \\ 2 & 3 & 5 \end{pmatrix}.$$

Use your result to find a nontrivial relation among the column vectors:

$$r \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + s \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix} + t \begin{pmatrix} 4 \\ 6 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

for some $r, s, t \in \mathbb{R}$ that are **not all zero**. [Hint: Relations among columns are not changed by row operations, so it is easier to find a relation among the columns of RREF(A).]

Solution. We perform Gaussian elimination to obtain RREF(A):

$$\begin{pmatrix} 1 & 2 & 4 \\ 3 & 4 & 6 \\ 2 & 3 & 5 \end{pmatrix} \begin{array}{l} 1 \\ 2 \\ 3 \end{array}$$

$$\begin{pmatrix} 1 & 2 & 4 \\ 0 & -2 & -6 \\ 0 & -1 & -3 \end{pmatrix} \begin{array}{l} 1 \\ 2 = 2 - 3\,1 \\ 3 = 3 - 2\,1 \end{array}$$

$$\begin{pmatrix} 1 & 2 & 4 \\ 0 & -2 & -6 \\ 0 & 0 & 0 \end{pmatrix} \begin{array}{l} 1 \\ 2 \\ 3 = 3 + \frac{1}{2}\,2 \end{array}$$

$$\begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix} \begin{array}{l} 1 \\ 2 = -\frac{1}{2}\,2 \\ 3 \end{array}$$

$$\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix} \begin{array}{l} 1 = 1 - 2\,2 \\ 2 \\ 3 \end{array}$$

In RREF(A) we observe that $-2(\text{column } 1) + 3(\text{column } 2) = (\text{column } 3)$:

$$-2\begin{pmatrix}1\\0\\0\end{pmatrix} + 3\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}-2\\3\\0\end{pmatrix}.$$

Since row operations preserve column relations, the same column relation must hold in A. And, indeed, we observe that

$$-2\begin{pmatrix}1\\3\\2\end{pmatrix} + 3\begin{pmatrix}2\\4\\3\end{pmatrix} = \begin{pmatrix}4\\6\\5\end{pmatrix}.$$

Finally, we can rewrite this relation in the desired form:

$$-2\begin{pmatrix}1\\3\\2\end{pmatrix} + 3\begin{pmatrix}2\\4\\3\end{pmatrix} - 1\begin{pmatrix}4\\6\\5\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}.$$

Bonus Discussion. Let B be some arbitrary 3×3 matrix and suppose that

$$\text{RREF}(B) = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}.$$

Then there is **no nontrivial column relation** in B or in RREF(B) because

$$r\begin{pmatrix}1\\0\\0\end{pmatrix} + s\begin{pmatrix}0\\1\\0\end{pmatrix} + t\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \quad \text{impliles } r = s = t = 0.$$

In other words, the columns of B must be **independent**.

Problem 4. The Solution Set of a Linear System is Flat. Consider the following system of m linear equations in n unknowns, where $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{a}_i = (a_{i1}, a_{i2}, \ldots, a_{in})$:

$$\begin{cases} \mathbf{a}_1 \bullet \mathbf{x} &= b_1, \\ \mathbf{a}_2 \bullet \mathbf{x} &= b_2, \\ & \vdots \\ \mathbf{a}_m \bullet \mathbf{x} &= b_m. \end{cases}$$

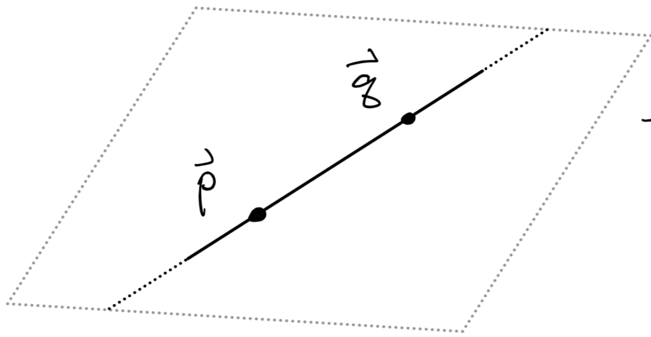
If $\mathbf{x} = \mathbf{p}$ and $\mathbf{x} = \mathbf{q}$ are any two points in the solution set, prove that every point of the line $\mathbf{x} = (1-t)\mathbf{p} + t\mathbf{q}$ is also in the solution set. [Hint: Assuming that $\mathbf{a}_i \bullet \mathbf{p} = b_i$ and $\mathbf{a}_i \bullet \mathbf{q} = b_i$ for all i, you are being asked to show that $\mathbf{a}_i \bullet [(1-t)\mathbf{p} + t\mathbf{q}] = b_i$ for all i.] Remark: This implies that the solution set is a d-plane in $\mathbb{R}^n$ for some d, or it is empty.

Proof. Let $\mathbf{x} = \mathbf{p}$ and $\mathbf{x} = \mathbf{q}$ be points of the solution set. By definition, this means that $\mathbf{a}_i \bullet \mathbf{p} = b_i$ and $\mathbf{a}_i \bullet \mathbf{q} = b_i$ for all $i = 1, \ldots, m$. Then for all $i = 1, \ldots, m$ and for all scalars t we observe that

$$\mathbf{a}_i \bullet [(1-t)\mathbf{p} + t\mathbf{q}] = (1-t)\mathbf{a}_i \bullet \mathbf{p} + t\mathbf{a}_i \bullet \mathbf{q} = (1-t)b_i + tb_i = b_i,$$

which means that the point $(1-t)\mathbf{p} + t\mathbf{q}$ is also in the solution set. $\square$

Bonus Discussion. If two points are in the solution set of a linear system then the whole line that they generate is also in the solution set. Here is a picture:



The solution set of a linear system is "flat."

Problem 5. Orthogonal Complement of a Subspace. A *d-dimensional subspace* of $\mathbb{R}^n$ is just a d-plane in $\mathbb{R}^n$ that contains the origin. If $\mathbf{u}_1, \ldots, \mathbf{u}_d \in \mathbb{R}^n$ are independent vectors (assume $d \leq n$) then their span is a d-dimensional subspace:

$$U = \{t_1\mathbf{u}_1 + \cdots + t_d\mathbf{u}_d : t_1, \ldots, t_d \in \mathbb{R}\}.$$

We define the *orthogonal complement* of this subspace as the set of vectors that are simultaneously perpendicular to every vector in U:[1]

$$U^\perp = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{u}_i \bullet \mathbf{x} = 0 \text{ for all } i\}.$$

Explain why $U^\perp$ is an $(n-d)$-dimensional subspace of $\mathbb{R}^n$. [Hint: The set $U^\perp$ is just the solution set of the linear equations $\mathbf{u}_i \bullet \mathbf{x} = 0$ for all i. We can express this system as a $d \times (n+1)$ matrix A. Since the rows of A are independent, we know that RREF(A) will have d pivots. So how many free variables does the system have?]

[Example: If $\mathbf{u}_1, \mathbf{u}_2 \in \mathbb{R}^3$ are independent vectors in 3-dimensional space, then $U \subseteq \mathbb{R}^3$ is the **plane** that they span and $U^\perp \subseteq \mathbb{R}^3$ is the **line** that is perpendicular to this plane, i.e., the line given by the cross product $\mathbf{u}_1 \times \mathbf{u}_2$. Hence we have $\dim U + \dim U^\perp = 2 + 1 = 3$ as expected.]

Proof. By definition, $U^\perp \subseteq \mathbb{R}^n$ is the solution set of the following linear system:

$$\begin{cases} \mathbf{u}_1 \bullet \mathbf{x} & = & 0, \\ \mathbf{u}_2 \bullet \mathbf{x} & = & 0, \\ & \vdots & \\ \mathbf{u}_d \bullet \mathbf{x} & = & 0. \end{cases}$$

We observe that $\mathbf{x} = \mathbf{0}$ is always a solution, so $U^\perp$ is some f-plane passing through the origin and we will prove that $f = n - d$. To do this, **we recall that the dimension of the solution set equals the number of free variables in the RREF of the system**. (This is why I used the letter f.) Since the row vectors are independent (indeed, we assumed that $\mathbf{u}_1, \ldots, \mathbf{u}_d$ are independent) there will be a pivot in each row of the RREF, hence there will be d pivot variables. Finally, we conclude that the number of free variables is

$$f = \#(\text{free variables}) = n - \#(\text{pivot variables}) = n - d.$$

$\square$

Remark: I realize that this is very abstract. See the course notes for discussion.

[1] Remark: If $\mathbf{x}$ is perpendicular to every basis vector $\mathbf{u}_i$, then it is also perpendicular to every linear combination $t_1\mathbf{u}_1 + \cdots + t_d\mathbf{u}_d$, hence it is perpendicular to every vector in the subspace U.