Math 210 **Fall 2020**
Homework 1 **Drew Armstrong**

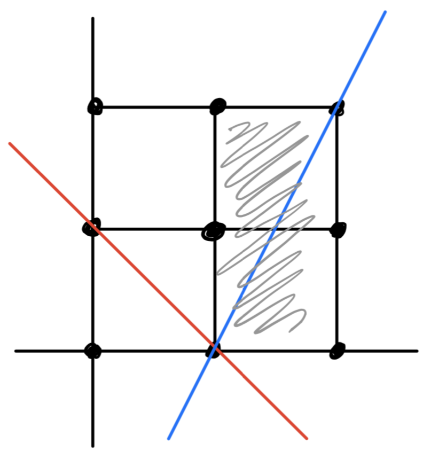
Problem 1. Coordinate Systems.

(a) Draw the 9 points (x, y) where $x, y \in \{0, 1, 2\}$. Draw the lines $y = -x+1$ and $y = 2x-2$. Shade the region where $1 \leq x \leq 2$ and $0 \leq y \leq 2$.

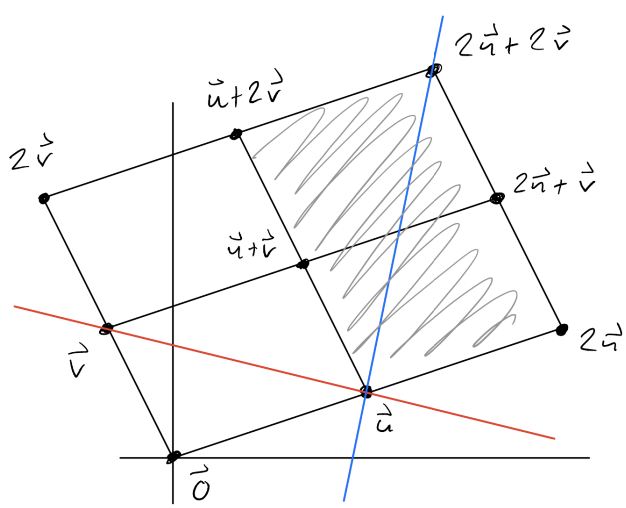
(b) Now let $\mathbf{u} = (3, 1)$ and $\mathbf{v} = (-1, 2)$. Draw the 9 points $x\mathbf{u} + y\mathbf{v}$ where $x, y \in \{0, 1, 2\}$. Draw the lines $\{x\mathbf{u} + y\mathbf{v} : y = -x + 1\}$ and $\{x\mathbf{u} + y\mathbf{v} : y = 2x - 2\}$. Shade the region $\{x\mathbf{u} + y\mathbf{v} : 1 \leq x \leq 2 \text{ and } 0 \leq y \leq 2\}$.

Remark: There was a typo in part (b). I have drawn the shaded region as I originally intended it, with $1 \leq x \leq 2$ instead of $0 \leq x \leq 1$. Your solution may look different.

Picture for (a): The red line is $y = -x + 1$ and the blue line is $y = 2x - 2$.



Picture for (b): The red line is $\{x\mathbf{u}+y\mathbf{v} : y = -x+1\}$ and the blue line is $\{x\mathbf{u}+y\mathbf{v} : y = 2x-2\}$.

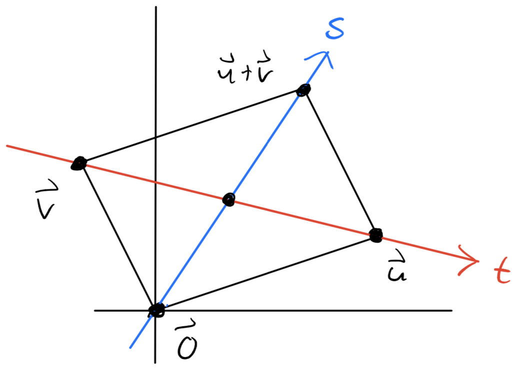


Problem 2. Midpoints. Consider the same points $\mathbf{u} = (3, 1)$ and $\mathbf{v} = (-1, 2)$.

(a) Draw the lines $t\mathbf{u} + (1 - t)\mathbf{v}$ and $s(\mathbf{u} + \mathbf{v})$. Show that $(\mathbf{u} + \mathbf{v})/2$ is the intersection point of these lines. Explain the geometric meaning.

(b) Now consider $\mathbf{w} = (2, 4)$. Draw the points $\mathbf{u}, \mathbf{v}, \mathbf{w}$. Also draw the point $(\mathbf{u} + \mathbf{v} + \mathbf{w})/3$ and explain its geometric meaning.

(a): The red line is $t\mathbf{u} + (1 - t)\mathbf{v} = t(3, 1) + (1 - t)(-1, 2) = (3t + (1 - t)(-1), t + (1 - t)2) = (4t - 1, 2 - t)$ and the blue line is $s(\mathbf{u} + \mathbf{v}) = s[(3, 1) + (-1, 2)] = s(2, 3) = (2s, 3s)$.

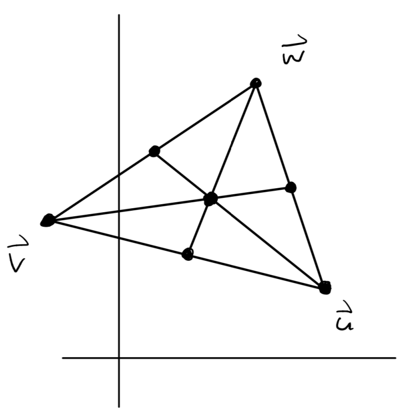


It is easy to check that the point $(\mathbf{u} + \mathbf{v})/2 = (1, 3/2)$ is on both of these lines, when $t = 1/2$ and $s = 1/2$. If we didn't already know this we could compute it as follows: The lines intersect when $(4t - 1, 2 - t) = (2s, 3s)$, i.e., when $4t - 1 = 2s$ and $2 - t = 3s$. Solve each equation for s to get $s = (4t - 1)/2$ and $s = (2 - t)/3$, then equate these expressions:

$$\begin{aligned}
(4t - 1)/2 &= (2 - t)/3 \\
3(4t - 1) &= 2(2 - t) \\
12t - 3 &= 4 - 2t \\
14t &= 7 \\
t &= 1/2.
\end{aligned}$$

Geometric meaning: The point $(\mathbf{u} + \mathbf{v})/2$ is the *midpoint* of $\mathbf{u}$ and $\mathbf{v}$. See course notes for further discussion.

(b): Here is the picture:



Geometric meaning: The point $(\mathbf{u}+\mathbf{v}+\mathbf{w})/3$ is the *centroid* (or the *center of mass*) of the three points $\mathbf{u}, \mathbf{v}, \mathbf{w}$. In the course notes we will show that this point is the intersection of the three lines that connect each vertex to the midpoint of the opposite side of the triangle.

Problem 3. The Angle Between Vectors. The general Pythagorean Theorem tells us that for any vectors $\mathbf{u}, \mathbf{v}$ in any number of dimensions we have $\mathbf{u} \bullet \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| \cos\theta$, where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$ measured tail-to-tail.

(a) Compute the angle between $\mathbf{u} = (3, 1)$ and $\mathbf{v} = (-1, 2)$.
(b) Now let $\mathbf{u}$ and $\mathbf{v}$ be any two vectors in 10 dimensional space satisfying $\mathbf{u} \bullet \mathbf{u} = 10$, $\mathbf{v} \bullet \mathbf{v} = 5$ and $\mathbf{u} \bullet \mathbf{v} = -1$. Compute the angle between $\mathbf{u}$ and $\mathbf{v}$.
(c) Now let $\mathbf{x}$ and $\mathbf{y}$ be any two vectors in 100 dimensional space satisfying $\mathbf{x} \bullet \mathbf{x} = \mathbf{y} \bullet \mathbf{y} = 1$ and $\mathbf{x} \bullet \mathbf{y} = 0$. Compute the angle between $\mathbf{u} = 3\mathbf{x} + \mathbf{y}$ and $\mathbf{v} = -\mathbf{x} + 2\mathbf{y}$.

(a): First we compute $\mathbf{u} \bullet \mathbf{u} = 3^2 + 1^2 = 10$, $\mathbf{v} \bullet \mathbf{v} = (-1)^2 + 2^2 = 5$ and $\mathbf{u} \bullet \mathbf{v} = 3(-1) + 1(2) = -1$. Then we compute

$$\cos\theta = \frac{\mathbf{u} \bullet \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{\mathbf{u} \bullet \mathbf{v}}{\sqrt{\mathbf{u} \bullet \mathbf{u}}\sqrt{\mathbf{v} \bullet \mathbf{v}}} = \frac{-1}{\sqrt{10}\sqrt{5}}.$$

It follows that $\theta = \arccos(-1/\sqrt{50}) \approx 98.13°$.

(b): Same answer as (a). The angle only depends on the dot products.

(c): First we compute

$$\mathbf{u} \bullet \mathbf{u} = (3\mathbf{x} + \mathbf{y}) \bullet (3\mathbf{x} + \mathbf{y}) = 9\mathbf{x} \bullet \mathbf{x} + 6\mathbf{x} \bullet \mathbf{y} + \mathbf{y} \bullet \mathbf{y} = 9 + 0 + 1 = 10,$$

then

$$\mathbf{v} \bullet \mathbf{v} = (-\mathbf{x} + 2\mathbf{y}) \bullet (-\mathbf{x} + 2\mathbf{y}) = \mathbf{x} \bullet \mathbf{x} - 4\mathbf{x} \bullet \mathbf{y} + 4\mathbf{y} \bullet \mathbf{y} = 1 + 0 + 4 = 5,$$

then

$$\mathbf{u} \bullet \mathbf{v} = (3\mathbf{x} + \mathbf{y}) \bullet (-\mathbf{x} + 2\mathbf{y}) = -3\mathbf{x} \bullet \mathbf{x} + 5\mathbf{x} \bullet \mathbf{y} 2\mathbf{y} \bullet \mathbf{y} = -3 + 0 + 2 = -1.$$

Again, the answer is the same as (a). Geometric meaning: We can think of $\mathbf{x}$ and $\mathbf{y}$ as an *orthonormal coordinate system*. Then the angle between vectors $a\mathbf{x} + b\mathbf{y}$ and $c\mathbf{x} + d\mathbf{y}$ in 100 dimensional space is the same as the angle between (a, b) and (c, d) in the Cartesian plane.

Problem 4. A Line in the Plane. Let $\mathbf{u} = (3, 1)$ and $\mathbf{v} = (-1, 2)$.

(a) Find the Cartesian equation of the line $t\mathbf{u} + (1 - t)\mathbf{v}$, i.e., in terms of x and y.
(b) Express this equation in the form $\mathbf{a} \bullet (x, y) = c$ for some vector $\mathbf{a}$ and some constant c.

Remark: This is the red line from Problem 1(b) and 2(a).

(a): There are many ways to do this. For example, the general point on this line has Cartesian coordinates

$$(x, y) = t\mathbf{u} + (1 - t)\mathbf{v} = t(3, 1) + (1 - t)(-1, 2) = (3t, t) + (t - 1, 2 - 2t) = (4t - 1, 2 - t),$$

i.e., $x = 4t - 1$ and $y = 2 - t$. Solving each of these for t gives $t = (x + 1)/4$ and $t = 2 - y$. Then equating these expressions gives

$$\begin{aligned} (x + 1)/4 &= 2 - y \\ x + 1 &= 8 - 4y \\ x + 4y &= 7. \end{aligned}$$

(b): We can express this equation in terms of the dot product:

$$\begin{pmatrix} 1 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} x \\ y \end{pmatrix} = 7.$$

Geometric meaning: This line is **perpendicular** to the vector $(1, 4)$. See Problem 5(b) below.

Problem 5. Parallel Lines. If a, b, c are constant then the equation $ax + by = c$ represents a line in the plane. This equation can also be expressed as

$$\mathbf{a} \bullet \mathbf{x} = c,$$

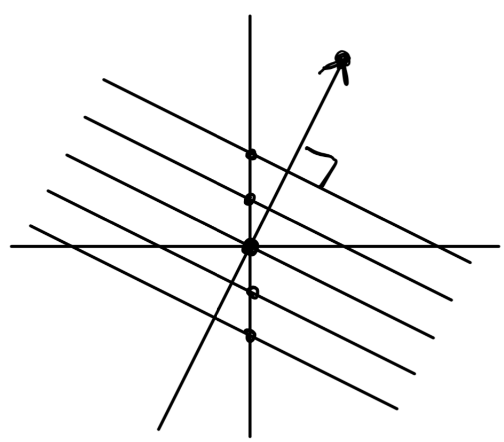
where $\mathbf{a} = (a, b)$ and $\mathbf{x} = (x, y)$.

(a) Draw the 5 lines $\mathbf{a} \bullet \mathbf{x} = c$ where $\mathbf{a} = (1, 2)$ and $c \in \{-2, -1, 0, 1, 2\}$.

(b) Now let $\mathbf{a} = (a, b)$ be an arbitrary nonzero vector in the plane and let c be an arbitrary constant. Prove that the line $\mathbf{a} \bullet \mathbf{x} = c$ is perpendicular to the line $t\mathbf{a}$. [Hint: For any two points $\mathbf{x}_1$ and $\mathbf{x}_2$ on the first line, show that the vector $\mathbf{x}_1 - \mathbf{x}_2$ is perpendicular to the vector $t\mathbf{a}$ for any t, i.e., that $(t\mathbf{a}) \bullet (\mathbf{x}_1 - \mathbf{x}_2) = 0$.]

It follows that any two lines of the form $\mathbf{a} \bullet \mathbf{x} = c_1$ and $\mathbf{a} \bullet \mathbf{x} = c_2$ are **parallel** (i.e., because they are both perpendicular to the line $t\mathbf{a}$).

(a): Each line has the form $(1, 2) \bullet (x, y) = x + 2y = c$ for some value of c. In other words, each line has the form $y = (-1/2)x + c/2$, with slope $-1/2$ and y-intercept $c/2$. Here are these lines when $c = \{-2, -1, 0, 1, 2\}$:



Observe that the lines are parallel. Indeed, they each have the same slope $-1/2$. Furthermore, observe that each of these lines is perpendicular to the vector $(1, 2)$ and the line $t(1, 2)$. We will explain this in part (b).

(b): Let $\mathbf{a} = (a, b) \in \mathbb{R}^2$ be any vector and let $c \in \mathbb{R}$ be any scalar. Then I claim that the line

$$ax + by = \mathbf{a} \bullet \mathbf{x} = c$$

is perpendicular to the line $t\mathbf{a} = t(a, b)$.

Proof. Let $\mathbf{x}_1 = (x_1, y_1)$ and $\mathbf{x}_2 = (x_2, y_2)$ be any two points on the first line. By definition this means that $ax_1 + by_1 = \mathbf{a} \bullet \mathbf{x}_1 = c$ and $ax_2 + by_2 = \mathbf{a} \bullet \mathbf{x}_2 = c$. Then it follows that

$$\mathbf{a} \bullet (\mathbf{x}_1 - \mathbf{x}_2) = \mathbf{a} \bullet \mathbf{x}_1 - \mathbf{a} \bullet \mathbf{x}_2 = c - c = 0.$$

In other words, the vector $\mathbf{x}_1 - \mathbf{x}_2$ is perpendicular to the vector $\mathbf{a}$. $\square$