The Growing Importance of Linear Algebra in Undergraduate Mathematics

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Linear algebra stands today as the epitome of accessible, yet powerful mathematical theory. Linear algebra has many appealing facets which radiate in different directions. In the 1960s, linear algebra was positioned to be the first real mathematics course in the undergraduate mathematics curriculum in part because its theory is so well structured and comprehensive, yet requires limited mathematical prerequisites. A mastery of finite vector spaces, linear transformations, and their extensions to function spaces is essential for a practitioner or researcher in most areas of pure and applied mathematics. Linear algebra is the mathematics of our modern technological world of complex multivariable systems and computers.

The advent of digital computers in the last forty years has eliminated the tedium of the extensive computations associated with linear systems. With computers, linear models such as linear programming and linear regression are now used to organize and optimize virtually all business activities from street-sweeping to market research to controlling oil refineries. While mathematical methods—principally calculus-based analysis—were once largely restricted to the physical sciences, tools of linear algebra find use in almost all academic fields and throughout modern society. The interaction with modern computation is especially appealing: previously, theory was needed to give analytic answers since explicit computation was hopelessly tedious; nowadays, theory is used to guide increasingly complex computations. As noted below, crucial developments in matrix theory and automated computations have occurred hand-in-hand ever since the term 'matrix' was coined by J. J. Sylvester in 1848.

There is an even more pervasive practical side of linear algebra. Stated in starkest terms, linear problems are solvable while nonlinear problems are not. Of course, some nonlinear problems with a small number of variables can be solved, but 99.99% of multivariable nonlinear problems can be solved only by recasting them as linear systems. For example, finite element and finite difference schemes for solving partial differential equations in the end rely on solving systems of n linear equations in n variables.

The theoretical status of linear algebra is as important as its applicability and its role in computation. Vector spaces and linear transformations are central themes of much of mathematics. The fact that differentiation is a linear operator lies at the heart of the power of calculus and differential equations. Of course, the very definition of the derivative concerns linearity: the slope of a tangent line—a local linear approximation—of a function. Fourier series arise from one orthogonal basis for the vector space of continuous functions. Most of functional analysis, especially topics such as Hilbert spaces and Fourier analysis, are part of the linear mathematics which grows naturally out of the concept of a vector space of functions introduced in sophomore linear algebra courses.

The pedagogical virtues of an introductory linear algebra course are just as impressive as the subject's usefulness and central role in higher analysis. Linear algebra gives a formal structure to analytic geometry and solutions to 2×2 and 3×3 systems of linear equations learned in high school. A vector space is the natural choice for a first algebraic system for students to study formally because its properties are all part of students' knowledge of analytic geometry. Unlike groups and fields, one can draw insightful pictures of elements in vector spaces. Linear transformations on (finite-dimensional) vector spaces also have concrete descriptions with matrices.

Matrix algebra generalizes the single-variable algebra of high school mathematics to give a very striking demonstration of the power of algebraic notation. For a simple example, the matrix equation $\mathbf{p}^{(10)} = A^{10}\mathbf{p}$ for the population $\mathbf{p}^{(10)}$ in the 10th period of a growth model presents a relationship between entries in $\mathbf{p}^{(10)}$ and in $\mathbf{p}$ that is far too complex to write out explicitly. Matrix algebra is the standard language for much of applied mathematics. For example, the least squares solution to a system of equations $A\mathbf{x} = \mathbf{b}$ is given by the matrix expression $(A^TA)^{-1}A^T\mathbf{b}$, and, building on this, the basic projection step in Karmarkar's algorithm for linear programming is given by the matrix expression $(I - A(A^TA)^{-1}A^T)$. (Note that in parallel with the 'low-level' and 'high-level' languages for programming computers, matrix algebra has a low-level, e.g., $c_{ij} = \Sigma a_{ik} b_{kj}$, and a high-level, e.g., $C = AB$, notational language.)

Linear algebra takes students' background in Euclidean space and formalizes it with vector space theory that builds on algebra and the geometric intuition developed in high school. Then this comfortable setting is shown to apply with unimagined generality, producing vector spaces of functions and more. Similarly, the easy-to-follow linear transformations on Euclidean space described through matrices generalize to linear operators on function spaces.

Linear algebra is also appealing because it is so powerful yet simple. There is a satisfying theoretical answer to almost any question a student can pose in linear algebra. The theory also leads directly to efficient computation. Even when a system of equations $A\mathbf{x} = \mathbf{b}$ has no solution (say, when A has more rows than columns and $\mathbf{b}$ is not in the range of A), linear algebra provides the pseudo-inverse to find a closest (least-squares) approximate solution. A first course in linear algebra contains beautiful classification theorems, such as the fact that every k-dimensional real vector space is isomorphic to R^k.

A further pedagogical strength of linear algebra is that it joins together methods and insights of geometry, algebra, and analysis; examples of these connections abound in the articles in this special issue of the *College Mathematics Journal*. This combination of contributing fields plus the powerful framework of vector spaces and linear transformations allows a sophomore course in linear algebra to define the ground rules for much of higher analysis, advanced geometry, statistics, operations research, and computational applied mathematics. For example, one of these ground rules is that it suffices to understand the action of a linear transformation on a set of basis functions and then let linearity do the rest.

Linear algebra really is a model for what a mathematical theory should be!

A Brief History of Linear Algebra and Matrix Theory

I would like to give a brief history of linear algebra and, especially, matrices. The subject is relatively young. Excluding determinants, its origins lie in the nineteenth century. Most interestingly, many of the advances in the nineteenth century came from non-mathematicians.

Matrices and linear algebra did not grow out of the study of coefficients of systems of linear equations, as one might guess. Arrays of coefficients led mathematicians to develop determinants, not matrices. Leibnitz, co-inventor of calculus, used determinants in 1693 about one hundred and fifty years before the study of matrices in their own right. Cramer presented his determinant-based formula for solving systems of linear equations in 1750. The first implicit use of matrices occurred in Lagrange's work on bilinear forms in the late 18th century. His objective was to characterize the maxima and minima of functions of several real variables. Besides requiring the first derivatives to be zero, he needed a condition on the matrix of second derivatives: the condition was positive definiteness or negative definiteness (although he did not use matrices).

Gauss developed Gaussian elimination around 1800, to solve least squares problems in celestial computations and later in geodetic computations. It should be noted that Chinese manuscripts from several centuries earlier have been found that explain how to solve a system of three equations in three unknowns by 'Gaussian' elimination. Gaussian elimination was for years considered part of the development of geodesy, not mathematics. Gauss-Jordan elimination's first appearance in print was in a handbook on geodesy by Wilhelm Jordan. The name Jordan in Gauss-Jordan elimination does not refer to the famous mathematician Camille Jordan, but rather to the geodesist Wilhelm Jordan. (Most linear algebra texts mistakenly identified Gauss-Jordan elimination with the mathematician Jordan, until a 1987 article in the *American Mathematical Monthly* by Athloen and McLaughlin [**1**], motivated by a historical talk by this author's father, A. W. Tucker, set the record straight.)

For matrix algebra to develop, one needed two things: the proper notation and the definition of matrix multiplication. Interestingly both these critical factors occurred at about the same time, around 1850, and in the same country, England. Except for Newton's invention of calculus, the major mathematical advances in the 1600s, 1700s and early 1800s were all made by Continental mathematicians, such as Bernoulli, Cauchy, Euler, Gauss, and Laplace. But in the mid-1800s, English mathematicians pioneered the study of various algebraic systems. For example, Augustus DeMorgan and George Boole developed the algebra of sets (Boolean algebra) in which symbols were used for propositions and abstract elements.

The introduction of matrix notation and the invention of the word matrix were motivated by attempts to develop the right algebraic language for studying determinants. In 1848, J. J. Sylvester introduced the term "matrix," the Latin word for womb, as a name for an array of numbers. He used womb, because he viewed a matrix as a generator of determinants. That is, every subset of k rows and k columns in a matrix generated a determinant (associated with the submatrix formed by those rows and columns).

Matrix algebra grew out of work by Arthur Cayley in 1855 about linear transformations. Given transformations,

$$T_1\colon\ x' = ax + by \qquad T_2\colon\ x'' = \alpha x' + \beta y'$$
$$\quad\ \ y' = cx + dy \qquad\qquad y'' = \gamma x' + \delta y',$$

he considered the transformation obtained by performing T_1 and then performing T_2.

$$T_2T_1\colon x'' = (\alpha a + \beta c)x + (\alpha b + \beta d)y$$
$$y'' = (\gamma a + \delta c)x + (\gamma b + \delta d)y.$$

In studying ways to represent this composite transformation, he was led to define matrix multiplication: the matrix of coefficients for the composite transformation T_2T_1 is the product of the matrix for T_2 times the matrix for T_1. Cayley went on to study the algebra of these compositions—matrix algebra—including matrix inverses. In his 1858 *Memoir on the Theory of Matrices*, Cayley gave the famous Cayley-Hamilton theorem: a square matrix satisfies its characteristic equation. The use of a single symbol A to represent the matrix of a transformation was essential notation of this new algebra. A link between matrix algebra and determinants was quickly established with the result $\det(AB) = \det(A)\det(B)$. But Cayley seemed to have realized that matrix algebra might grow to overshadow the theory of determinants. He wrote, "There would be many things to say about this theory of matrices which should, it seems to me, precede the theory of determinants."

It is a curious sidelight to this discussion that another prominent English mathematician of this time was Charles Babbage who built the first modern calculating machine. Abstracting the mechanics of computation as well as its algebraic structure and notation (and DeMorgan's work on the algebra of sets which would later be crucial in computer science) seemed to be all part of the same general intellectual pattern in England in the mid-nineteenth century.

Mathematicians also tried to develop an algebra of vectors but there was no natural general definition for the product of two vectors. The first vector algebra, involving a noncommutative vector product, was proposed by Hermann Grassmann's first *Ausdehnungslehre* (1844). This text also introduced column-row products, what are now called simple matrices or rank-one matrices (formed by matrix multiplication of a column vector times a row vector). The famous treatise on vector analysis by the late 19th-century American mathematical physicist J. Willard Gibbs developed vector and matrix theory further [**6**], including representations of general matrices, which he called dyadics, as a sum of simple matrices, which Gibbs called dyads. Later the physicist P. A. M. Dirac introduced the term "bra-ket" for what we now call the scalar product of a "bra" (row) vector times a "ket" (column) vector, while the term "ket-bra" referred to the product of "ket" (column) times "bra" (row), yielding what we call here a simple matrix. (Physicists in the 20th century developed the convention of assuming any vector was implicitly a column vector with a row vector represented as the transpose of a column vector.)

Matrices remained closely associated with linear transformations and, from the theoretical viewpoint, were by 1900 just a finite-dimensional subcase of an emerging general theory of linear transformations. Matrices were also viewed as a powerful notation, but after an initial spurt of interest in the nineteenth century were little studied in their own right. More attention was paid to vectors, which are basic mathematical elements of physics as well as many areas of mathematics. The modern definition of a vector space was introduced by Peano in 1888. Abstract vector spaces, whose elements were functions or linear transformations, soon followed.

Interest in matrices, with emphasis on their numerical analysis, re-emerged after World War II with the development of modern digital computers. In 1947, John von Neumann and Herman Goldstine introduced condition numbers in analyzing round-off error. Alan Turing, the other giant (with von Neumann) in the development of stored-program computers, introduced the *LU* decomposition of a matrix in 1948. The usefulness of the *QR* decomposition was realized a decade later. The most important contributor in this effort was J. H. Wilkinson, who, among other achievements, showed the stability of Gaussian elimination, still the best way known to solve a system of linear equations. See **[4]**, **[9]** for information about the foundations of numerical linear algebra.

Linear Algebra in the Undergraduate Curriculum

Although linear algebra is now solidly established in the lower-division collegiate mathematics curriculum, it is important to remember this role is fairly recent. The other topics in the beginning mathematics sequence—single-variable calculus, differential equations, and multivariate calculus—have been part of the collegiate curriculum for two hundred years, while linear algebra, in its current vector-space format, first appeared in basic graduate texts in the mid-1940s and first appeared in undergraduate texts, as part of abstract algebra, in the late-1950s. The first lower-division text with vector spaces was Kemeny, Snell, Thompson and Mirkel's 1959 *Finite Mathematical Structures*, which combined vector-space theory with matrix-based applications, such as Markov chains and linear programming.

Before the publication of the landmark 1941 text, *Modern Algebra*, by Birkhoff and MacLane, 20th-century algebra texts focused on algebraic solutions to polynomial equations (leading to algebraic geometry) and related matters. The fundamental theorem of algebra, proved by Gauss, that every polynomial over the complex numbers has a root, illustrates this 'old' algebra. The linear algebra then studied grew out of Lagrange's work on bilinear forms mentioned above. Matrices were barely mentioned. The 'modern' algebra, pushed by Emmy Noether and disciples of hers, such as E. Artin and B. van der Waerden, stressed the intrinsic structure of algebraic systems. In this setting, linear algebra was about the algebraic structure of linear maps (linear transformations). Matrices were viewed as having limited intrinsic value, since the matrix used to represent a finite-dimensional linear transformation depended on the choice of basis. (For more, see **[7]**).

The text of Kemeny et al. presented the two sides of linear algebra, vector spaces and matrix applications. When linear algebra was widely adopted in the 1960s in the lower-division mathematics curriculum, the course followed the vector space syllabus of the 1965 CUPM Recommendations for a General Collegiate Mathematics Curriculum, growing out to the 'modern' algebra approach of Noether. Such a focus has two very important functions: giving students a very accessible, geometrically based theory whose study serves as a preparation to more abstract, upper-division mathematics courses; and providing a framework for a modern vector-space approach to a sequel course in multivariate calculus. At the same time, the use of matrix-based models like those in Kemeny et al. (e.g., Markov chains) has exploded with the accessibility of digital computers. Some argue that linear algebra and associated matrix models are as widely used a mathematical tool as calculus and would like to see linear algebra taught for a broader audience with the sort of practical point of view used to teach calculus.

An example of the sort of topic that might get more attention in a matrix-based course would be the 'asymptotic' behavior of inverses. In a theoretical framework, either an $n \times n$ matrix has an inverse A or does not; if it does then the columns of

A are linearly independent and $dim(Range(A)) = n$, etc. The matrix-based course would examine algebraically, analytically, and geometrically what happens when a slight perturbation in some coefficients of **A** causes the inverse to 'blow up' and cease to exist. One would introduce the concept of the condition number of A to measure how close A is to blowing up.

Another example involves the very definition of matrix multiplication. Vector-space-based texts use matrix multiplication rarely (just to compose two linear maps and in the definition of similarity). Thus matrices are just a tool and the entry-by-entry definition $c_{ij} = \Sigma a_{ik} b_{kj}$ is about all that is said about computing the matrix product $C = AB$. However, a matrix-based text would focus on rows and columns, rather than entries. If $\mathbf{a}_j$ denotes the j-th column of matrix A and $\mathbf{a}'_i$ denotes the i-th row of A, then we have that $c_{ij} = \mathbf{a}'_i \mathbf{b}_j$ and $C = \Sigma \mathbf{a}_k * \mathbf{b}'_k$ (where $*$ denotes matrix multiplication of a column times a row). Also $C = AB$ can be defined column-by-column with $\mathbf{c}_j = A\mathbf{b}_j$ or row-by-row with $\mathbf{c}'_i = \mathbf{a}'_i B$.

An NSF-sponsored workshop on college linear algebra at Williamsburg in August 1990 recommended giving more attention to matrix algebra and its applications, while endorsing the current level of theory and rigor. A recent report of the CUPM Subcommittee on the Mathematics Major finds merit in both approaches (provided the vector space theory is covered in a subsequent upper-division course if the first linear algebra course emphasizes matrix methods). Not surprisingly, linear algebra courses offered outside mathematics departments concentrate mainly on matrix methods and models, although these 'users' courses also cover vector space foundations. In support of matrix-based applications, it is argued that most mathematics majors do not go to graduate school, but work in business or teach in secondary schools, where an applied, matrix-based focus is more useful. On the other hand, since calculus is taught as a service course with little theory, a balancing theory orientation to linear algebra appears vital for mathematics majors advancing to courses in abstract algebra and analysis.

In smaller institutions where only one course in linear algebra can realistically be offered, the challenge is to try to find a middle ground blending vector spaces and matrix methods and at a level that does not scare off users and yet smooths the transition for mathematics majors to advanced courses. More broadly, practitioners and theoreticians should work together in search of common ground and an understanding of each others' interests.

This author is personally a bit frustrated that calculus gets all of the first year and half of the second year of the lower-divison core mathematics sequence. Hopefully early in the next century, there will be a 'redistricting' of the lower-division mathematics sequence and linear algebra will get equal time with calculus.

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To Each His Own Space

In teaching linear algebra I have used the following quotation, usually soon after I have shown that the way we multiply two matrices is forced on us by the definition of the composition of two linear transformations.

"Me, I prefer pure thought. According to my subjective judgment calculating is on a lower intellectual (artistic?) level than thinking: it is more like the tenpenny nails of the carpenter than the blueprints of the architect. As far as I am concerned *matrices are mysteries and linear transformations light the way*. Matrix multiplication is an algorithm—but do you remember the proof that it is associative? It is a computational mess, and if you don't know the other way to get at it, by "pure thought", you don't understand it. As a student I was exposed to the matrix theory in Bôcher's regrettable old book, and I never understood what was going on till I heard von Neumann lecture about infinite-dimensional Hilbert spaces."

P. R. Halmos, Pure thought is better yet..., *College Mathematics Journal* 16 (1985) p. 14)

However, I've always felt that this quotation was a bit one-sided. Now I can complement it with one by Irving Kaplansky, a good friend of Paul Halmos.

"We (*Halmos and Kaplansky*) share a love of linear algebra. I think it is our conviction that we'll never understand infinite-dimensional operators properly until we have a decent mastery of finite matrices. And we share a philosophy about linear algebra: we think basis-free, we write basis-free, but when the chips are down we close the office door and compute with matrices like fury."

Paul Halmos, Celebrating 50 Years of Mathematics, John H. Ewing and F. W. Gehring (Eds.), Springer-Verlag, NY, 1991, p. 88.

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