HW6 Notes

We have finished discussing "least squares regression", which is one of the most common applications of Linear Algebra.

There is one more application of Linear Algebra I want to discuss before sending you out into the world. I'll call it

"spectral analysis"

and I'll also introduce this topic with an example.

Motivating Example: You may have heard of the "Fibonacci Sequence"

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, etc.

If we write f_n for the n^{th} Fibonacci number then the sequence is defined by the "initial conditions"

$$f_0 = 0 \quad \& \quad f_1 = 1$$

and the "recurrence equation"

$$f_{n+2} = f_{n+1} + f_n \quad \text{for all } n \geq 0.$$

For example, we have

$$\begin{aligned} f_2 &= f_1 + f_0 = 1 + 0 = 1 \\ f_3 &= f_2 + f_1 = 1 + 1 = 2 \\ f_4 &= f_3 + f_2 = 2 + 1 = 3 \\ f_5 &= f_4 + f_3 = 3 + 2 = 5, \text{ etc.} \end{aligned}$$

Our goal today is to find a "closed formula" for the n^{th} Fibonacci number:

$$f_n = ?$$

The answer is very hard to guess, but we can compute it rather easily using a trick and some Linear Algebra. The trick is to rewrite the recurrence equation as a system of two linear equations

$$\begin{cases} f_{n+2} = f_{n+1} + f_n \\ f_{n+1} = f_{n+1} \end{cases}$$

The second equation looks quite useless but it's not because it allows us to express the recurrence as a matrix equation

$$(*) \quad \begin{pmatrix} f_{n+2} \\ f_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} f_{n+1} \\ f_n \end{pmatrix}.$$

To save some space we will introduce the notations

$$\vec{f}_n = \begin{pmatrix} f_{n+1} \\ f_n \end{pmatrix} \quad \& \quad T = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},$$

Then we can express the initial conditions and the recurrence as follows:

- $\vec{f}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
- $\vec{f}_{n+1} = T\vec{f}_n$ for all $n \geq 0$.

Now we're ready to apply Linear Algebra. By computing the first few vectors $\vec{f}_n$,

$$\vec{f}_1 = T\vec{f}_0$$

$$\vec{f}_2 = T\vec{f}_1 = T(T\vec{f}_0) = (TT)\vec{f}_0 = T^2\vec{f}_0$$

$$\vec{f}_3 = T\vec{f}_2 = T(T^2\vec{f}_1) = (TT^2)\vec{f}_0 = T^3\vec{f}_0$$

we see that the n^{th} vector is given by

$$\vec{f}_n = T^n\vec{f}_0$$

$$\begin{pmatrix} f_{n+1} \\ f_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

and we really only care about the 2nd entry of this vector, which is the n^{th} Fibonacci number f_n.

Great, now let's compute it. HOW?

This is where the "spectral analysis" comes in. We make the following fundamental definition.

☆ Definition: We will say that a number λ is an <u>eigenvalue</u> for the matrix T if there exists a nonzero vector $\vec{x} \neq \vec{0}$ such that

$$\boxed{T\vec{x} = \lambda\vec{x}}$$

and in this case we will say that $\vec{x}$ is a <u>λ-eigenvector</u> of T. ///

The whole reason for this definition is the following observation: If $\vec{x}$ is a λ-eigenvector for T then we have

$$\begin{aligned} T^2\vec{x} &= T(T\vec{x}) \\ &= T(\lambda\vec{x}) \\ &= \lambda(T\vec{x}) \\ &= \lambda(\lambda\vec{x}) = \lambda^2\vec{x}, \end{aligned}$$

$$\begin{aligned} T^3\vec{x} &= T(T^2\vec{x}) \\ &= T(\lambda^2\vec{x}) \\ &= \lambda^2(T\vec{x}) \\ &= \lambda^2(\lambda\vec{x}) = \lambda^3\vec{x}, \end{aligned}$$

and in general we have

$$\boxed{T^n\vec{x} = \lambda^n\vec{x}}.$$

So if $\vec{f}_0 = (1,0)$ were an eigenvector of T we would be done. Unfortunately it's not:

$$T\vec{f}_0 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

which is not of the form $\lambda\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. But that's okay. Here's the big idea.

☆ Idea of Spectral Analysis:

If we can express the initial condition $\vec{f}_0$ as a linear <u>combination of eigenvectors</u> for the transition matrix T, then we will be done.

Indeed, suppose that $\vec{u}$ & $\vec{v}$ are eigenvectors for T with

$$T\vec{u} = \lambda \vec{u} \qquad \& \qquad T\vec{v} = \mu \vec{v},$$

for some eigenvalues λ & μ and suppose that we can write

$$\vec{f_0} = a\vec{u} + b\vec{v}$$

for some numbers a & b. Then we will have

$$\begin{aligned} T^n \vec{f_0} &= T^n(a\vec{u} + b\vec{v}) \\ &= a(T^n\vec{u}) + b(T^n\vec{v}) \\ &= a\lambda^n\vec{u} + b\mu^n\vec{v} \end{aligned}$$

and the problem will be solved! Thus we have reduced the problem to:

- finding enough eigenvectors for T
- expressing the initial condition $\vec{f_0}$ in terms of them.

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...continued

Right now I am introducing the idea of "spectral analysis" through a motivational example.

Recall the Fibonacci numbers

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots .$$

These are defined by initial conditions

$$f_0 = 0 \ \& \ f_1 = 1 ,$$

and by the recurrence equation

$$f_{n+2} = f_{n+1} + f_n \quad \text{for } n \geq 0 .$$

Our goal is to "solve" this recurrence, i.e., to find a "closed formula" for the n^{th} Fibonacci number. The answer is very hard to guess so it is preferable to develop a mechanical technique.

To do this we will define the vectors

$$\vec{f}_n := \begin{pmatrix} f_{n+1} \\ f_n \end{pmatrix}$$

consisting of two consecutive Fibonacci numbers and then observe that the initial conditions and recurrence can be rewritten in terms of matrix algebra as

$$\vec{f}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \& \quad \vec{f}_{n+1} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \vec{f}_n \text{ for } n \geq 0.$$

If we define $T = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ then we can solve explicitly for $\vec{f}_n$:

$$\boxed{\vec{f}_n = T^n \vec{f}_0}$$

Now the whole problem is to investigate the powers T^n of the matrix T, and the key tools for doing this are called eigenvalues & eigenvectors;

☆ Consider a nonzero vector $\vec{x} \neq \vec{0}$. We say that $\vec{x}$ is an eigenvector for the matrix T if there exists some number λ such that

$$T\vec{x} = \lambda \vec{x} .$$

In this case we say that λ is the eigenvalue corresponding to the eigenvector $\vec{x}$. (Sometimes we say that $\vec{x}$ is a "λ-eigenvector" of T.) ///

☆ The idea of spectral analysis is to express the initial condition $\vec{f}_0$ as a linear combination of eigenvectors for the transition matrix.

I'll show you how to compute the eigenvectors in a bit. Right now let me just tell you the answer.

If we define the numbers

$$\varphi_1 = \frac{1+\sqrt{5}}{2} \quad \& \quad \varphi_2 = \frac{1-\sqrt{5}}{2}$$

then I claim [just believe me] that.

- $$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \varphi_1 \\ 1 \end{pmatrix} = \varphi_1 \begin{pmatrix} \varphi_1 \\ 1 \end{pmatrix}$$

- $$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \varphi_2 \\ 1 \end{pmatrix} = \varphi_2 \begin{pmatrix} \varphi_2 \\ 1 \end{pmatrix}$$

- $$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} \varphi_1 \\ 1 \end{pmatrix} - \frac{1}{\sqrt{5}}\begin{pmatrix} \varphi_2 \\ 1 \end{pmatrix}$$

And then the answer to our problem is immediate. We have

$$\begin{aligned}
\begin{pmatrix} f_{n+1} \\ f_n \end{pmatrix} = \vec{f}_n = T^n \vec{f}_0 \\
&= T^n \left(\frac{1}{\sqrt{5}}\begin{pmatrix} \varphi_1 \\ 1 \end{pmatrix} - \frac{1}{\sqrt{5}}\begin{pmatrix} \varphi_2 \\ 1 \end{pmatrix} \right) \\
&= \frac{1}{\sqrt{5}} T^n \begin{pmatrix} \varphi_1 \\ 1 \end{pmatrix} - \frac{1}{\sqrt{5}} T^n \begin{pmatrix} \varphi_2 \\ 1 \end{pmatrix} \\
&= \frac{1}{\sqrt{5}} \varphi_1^n \begin{pmatrix} \varphi_1 \\ 1 \end{pmatrix} - \frac{1}{\sqrt{5}} \varphi_2^n \begin{pmatrix} \varphi_2 \\ 1 \end{pmatrix}.
\end{aligned}$$

Then comparing the second entry in each vector gives us the desired formula for the n^{th} Fibonacci number:

$$f_n = \frac{1}{\sqrt{5}}\varphi_1^n - \frac{1}{\sqrt{5}}\varphi_2^n$$

$$= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n \quad (!)$$

I consider this formula pretty amazing because it doesn't even look like a whole number. Let's check a couple of cases:

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^0 - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^0$$

$$= \frac{1}{\sqrt{5}}\cdot 1 - \frac{1}{\sqrt{5}}\cdot 1 = 0 = f_0 \quad \checkmark$$

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^1 - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^1$$

$$= \frac{1}{2\sqrt{5}}\left[(\cancel{1}+\sqrt{5}) - (\cancel{1}-\sqrt{5})\right]$$

$$= \frac{1}{\cancel{2\sqrt{5}}}\left[\cancel{2\sqrt{5}}\right] = 1 = f_1 \quad \checkmark.$$

OK, that's good enough for me :) .

What remains to do?

I need to show you how to <u>compute</u> the eigenvalues & eigenvectors of a matrix if you don't know them already. Actually, this is pretty hard in general so I'll just show you how to do it for 2×2 matrices.

So let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and suppose that λ is eigenvalue of A. This means that there exists a vector $\vec{x} \neq \vec{0}$ such that

$$\begin{aligned} A\vec{x} &= \lambda \vec{x} \\ A\vec{x} &= \lambda I_2 \vec{x} \\ A\vec{x} - \lambda I_2 \vec{x} &= \vec{0} \\ (A - \lambda I_2)\vec{x} &= \vec{0}. \end{aligned}$$

Since $\vec{x} \neq \vec{0}$ this equation tells me that the matrix $A - \lambda I_2$ has a non-trivial column relation, so it is not <u>invertible</u>.

↓

If $A - \lambda I_2$ were invertible then its inverse would be given by the formula

$$(A - \lambda I_2)^{-1} = \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right]^{-1}$$

$$= \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}^{-1}$$

$$= \frac{1}{(a-\lambda)(d-\lambda) - bc} \begin{pmatrix} d-\lambda & -b \\ -c & a-\lambda \end{pmatrix}.$$

But since we know that $A - \lambda I_2$ is not invertible, it must be the case that

$$(a-\lambda)(d-\lambda) - bc = 0$$
$$ad - a\lambda - d\lambda + \lambda^2 - bc = 0$$

$$\boxed{\lambda^2 - (a+d)\lambda + (ad - bc) = 0}$$

This is called the characteristic equation of the matrix A. Its solutions λ are precisely the eigenvalues of A, and we can compute them using the quadratic formula:

$$\lambda = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2} \quad (*)$$

After finding the eigenvalues from (*) it is easy to find the corresponding eigenvectors by solving the linear system

$$(A - \lambda I_2)\vec{x} = \vec{0}$$

for each eigenvalue λ.

Consider a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Recall that we say $\vec{x} = (x, y)$ is an eigenvector if

- $\vec{x} \neq \vec{0}$

- There exists a number λ (called the eigenvalue of $\vec{x}$) such that

$$A\vec{x} = \lambda \vec{x}.$$

In this case we can write

$$A\vec{x} = \lambda I_2 \vec{x}$$

$$A\vec{x} - \lambda I_2 \vec{x} = \vec{0}$$

$$(*) \qquad (A - \lambda I_2)\vec{x} = \vec{0}.$$

Since $\vec{x} \neq \vec{0}$ this implies that the matrix $A - \lambda I_2$ is not invertible. [If it were invertible then we could multiply both sides of $(*)$ by the inverse $(A - \lambda I_2)^{-1}$ to get

$$\cancel{(A-\lambda I_2)^{-1}}(A-\lambda I_2)\vec{x} = (A-\lambda I_2)^{-1}\vec{0}$$
$$\vec{x} = \vec{0},$$

which is a contradiction.] Since $A-\lambda I_2$ is not invertible we know that its determinant is zero,

$$0 = \det(A-\lambda I_2)$$
$$= \det\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right)$$
$$= \det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}$$
$$= (a-\lambda)(d-\lambda) - bc$$
$$= \lambda^2 - (a+d)\lambda + (ad-bc),$$

which allows us to compute the eigenvalues of A using the quadratic formula:

$$\lambda = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}.$$

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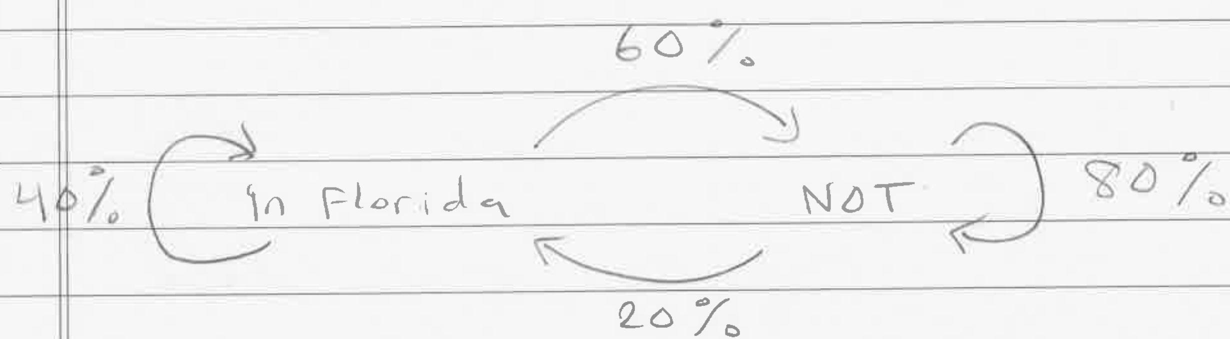
Application: Consider a certain population of bears. Let f_n denote the number of bears inside Florida at year n and let g_n denote the number of bears outside Florida. Suppose that the bears migrate according to the following pattern

$$f_{n+1} = 0.4 f_n + 0.2 g_n$$
$$g_{n+1} = 0.6 f_n + 0.8 g_n.$$

$$\begin{pmatrix} f_{n+1} \\ g_{n+1} \end{pmatrix} = \begin{pmatrix} 0.4 & 0.2 \\ 0.6 & 0.8 \end{pmatrix} \begin{pmatrix} f_n \\ g_n \end{pmatrix}.$$

We can also express this information with a transition diagram



Among the bears in Florida, 40% will stay in Florida next year and 60% will leave. Among the bears not in Florida, 20% will come to Florida next year and 80% will stay away. [This is a very simple model because it assumes that no bears are born or die.].

Our goal is to investigate the long term behavior of the bears:

$$\begin{pmatrix} f_n \\ g_n \end{pmatrix} \longrightarrow \text{?} \quad \text{as } n \longrightarrow \infty.$$

To do this we will compute the eigenvalues & eigenvectors of the transition matrix

$$T = \begin{pmatrix} 0.4 & 0.2 \\ 0.6 & 0.8 \end{pmatrix}.$$

The characteristic equation is

$$(0.4-\lambda)(0.8-\lambda) - (0.2)(0.6) = 0.$$

$$\lambda^2 - (1.2)\lambda + (0.4)(0.8) - (0.2)(0.6) = 0.$$

$$\lambda^2 - 1.2\lambda + 0.32 - 0.12 = 0$$

$$\lambda^2 - 1.2\lambda + 0.2 = 0.$$

$$10\lambda^2 - 12\lambda + 2 = 0.$$

So the eigenvalues are

$$\lambda = \left(12 \pm \sqrt{144-80}\right)/20$$
$$= \left(12 \pm \sqrt{64}\right)/20$$
$$= \left(12 \pm 8\right)/20$$
$$= 1 \text{ or } 0.2.$$

Now let's compute the eigenvectors. For eigenvalue $\lambda = 1$ we have

$$(T - 1I_2)\vec{x} = \vec{0} \longrightarrow \begin{pmatrix} 0.4-1 & 0.2 \\ 0.6 & 0.8-1 \end{pmatrix}\vec{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\longrightarrow \left(\begin{array}{cc|c} -0.6 & 0.2 & 0 \\ 0.6 & -0.2 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} -0.6 & 0.2 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cc|c} 1 & -1/3 & 0 \\ 0 & 0 & 0 \end{array}\right) \longrightarrow \begin{cases} x - \frac{1}{3}y = 0 \\ 0 = 0. \end{cases}$$

Let y be free. Then we have

$$\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/3\, y \\ y \end{pmatrix} = y \begin{pmatrix} 1/3 \\ 1 \end{pmatrix}.$$

Or, letting $t = \frac{1}{3} y$ gives

$$\vec{x} = t \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

These are the eigenvectors with eigenvalue 1.

For eigenvalue $\lambda = 0.2$ we have

$$(T - 0.2 I_2)\vec{x} = \vec{0} \longrightarrow \begin{pmatrix} 0.4 - 0.2 & 0.2 \\ 0.6 & 0.8 - 0.2 \end{pmatrix} \vec{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\longrightarrow \left(\begin{array}{cc|c} 0.2 & 0.2 & 0 \\ 0.6 & 0.6 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$\longrightarrow \begin{cases} x + y = 0 \\ 0 = 0. \end{cases}$$

Let y be free. Then we have

$$\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ y \end{pmatrix} = y \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

These are the eigenvectors of eigenvalue 0.2.
In summary, we have

$$\boxed{T\begin{pmatrix}1\\3\end{pmatrix} = 1\begin{pmatrix}1\\3\end{pmatrix} \quad \& \quad T\begin{pmatrix}-1\\1\end{pmatrix} = (0.2)\begin{pmatrix}-1\\1\end{pmatrix}}$$

These formulas give us all the information we need to analyze the system.

For example, suppose we start in year zero with 100 bears in Florida and 0 outside:

$$\begin{pmatrix}f_0\\g_0\end{pmatrix} = \begin{pmatrix}100\\0\end{pmatrix}.$$

To express this in terms of eigenvectors suppose that

$$a\begin{pmatrix}1\\3\end{pmatrix} + b\begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}100\\0\end{pmatrix},$$

$$\begin{pmatrix}1 & -1\\3 & 1\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}100\\0\end{pmatrix}.$$

Then we can solve for a & b:

$$\left(\begin{array}{cc|c} 1 & -1 & 100 \\ 3 & 1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & -1 & 100 \\ 0 & 4 & -300 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cc|c} 1 & -1 & 100 \\ 0 & 1 & -75 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & 0 & 25 \\ 0 & 1 & -75 \end{array}\right)$$

$$\rightarrow \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 25 \\ -75 \end{pmatrix} \rightarrow \begin{pmatrix} 100 \\ 0 \end{pmatrix} = 25 \begin{pmatrix} 1 \\ 3 \end{pmatrix} - 75 \begin{pmatrix} -1 \\ 1 \end{pmatrix},$$

Finally, we obtain the number of bears inside & outside Florida at year n:

$$\begin{pmatrix} f_n \\ g_n \end{pmatrix} = T^n \begin{pmatrix} f_0 \\ g_0 \end{pmatrix}$$

$$= T^n \left[25 \begin{pmatrix} 1 \\ 3 \end{pmatrix} - 75 \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right]$$

$$= 25\, T^n \begin{pmatrix} 1 \\ 3 \end{pmatrix} - 75\, T^n \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$= 25\,(1)^n \begin{pmatrix} 1 \\ 3 \end{pmatrix} - 75\,(0.2)^n \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 25 + 75(0.2)^n \\ 75 - 75(0.2)^n \end{pmatrix}.$$

In other words,

$$f_n = 25 + 75(0.2)^n$$
$$g_n = 75 - 75(0.2)^n.$$

And now we can see clearly what happens as $n \to \infty$. Since $(0.2)^n \to 0$ as $n \to \infty$ we have

$$\begin{pmatrix} f_n \\ g_n \end{pmatrix} \to \begin{pmatrix} 25 \\ 75 \end{pmatrix} \quad \text{as } n \to \infty.$$

In the long term there will be 25 bears inside Florida & 75 bears outside.

Conclusion of Fibonacci Example

For example, consider the Fibonacci matrix

$$T = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

We will denote the eigenvalues by

$$\varphi_1 = \frac{1+\sqrt{5}}{2} \approx 1.61 \qquad \& \qquad \varphi_2 = \frac{1-\sqrt{5}}{2} \approx -0.61$$

This one is called the "golden ratio"

On the HW you computed the eigenvectors

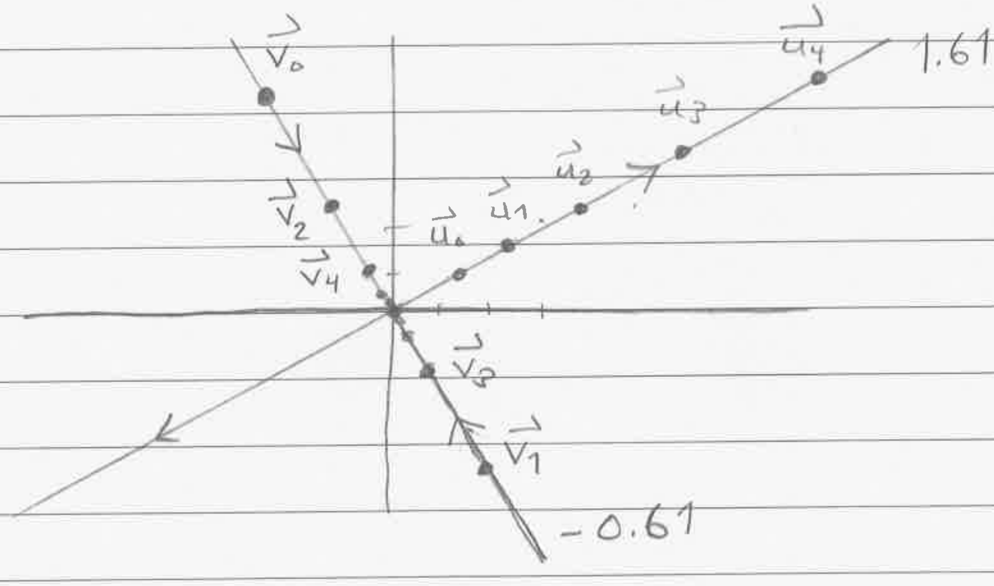
$$T\begin{pmatrix}\varphi_1\\1\end{pmatrix} = \varphi_1\begin{pmatrix}\varphi_1\\1\end{pmatrix} \quad \& \quad T\begin{pmatrix}\varphi_2\\1\end{pmatrix} = \varphi_2\begin{pmatrix}\varphi_2\\1\end{pmatrix}.$$

Actually it's more correct to talk about "eigendirections" or "eigenlines" because any scalar multiple of an eigenvector is still an eigenvector with the same eigenvalue. For example, for any number t we have

$$T\left(t\begin{pmatrix}\varphi_1\\1\end{pmatrix}\right) = t\,T\begin{pmatrix}\varphi_1\\1\end{pmatrix}$$

$$= t\,\varphi_1\begin{pmatrix}\varphi_1\\1\end{pmatrix} = \varphi_1\left(t\begin{pmatrix}\varphi_1\\1\end{pmatrix}\right).$$

Here are the two "eigenlines" for the matrix T:



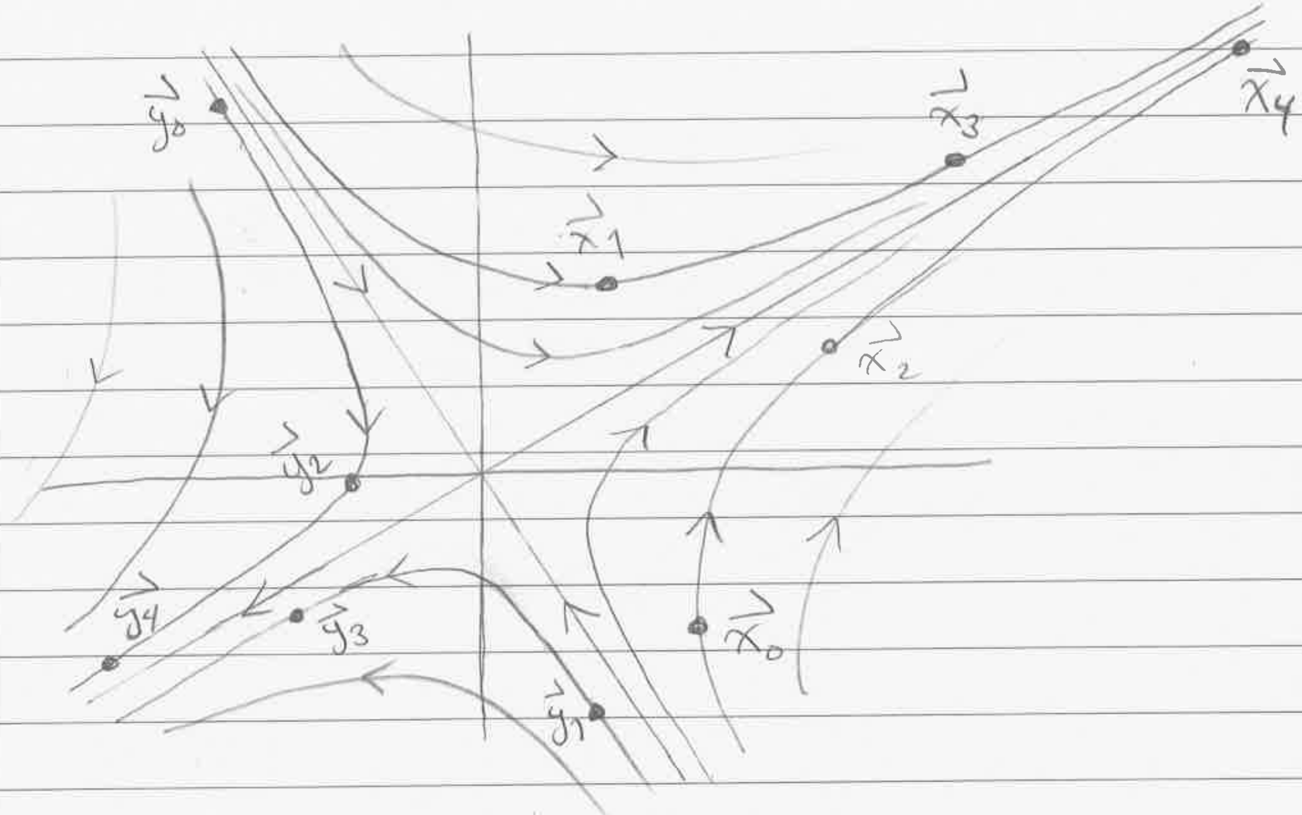
I have labeled each eigenline by its eigenvalue. The arrows indicate that one eigenline tends to expand ($|\varphi_1| > 1$) while the other tends to contract ($|\varphi_2| < 1$).

I have also drawn two sample trajectories for some initial conditions $\vec{u}_0$ & $\vec{v}_0$ in the eigenlines. That is we define

$$\vec{u}_n = T\,\vec{u}_{n-1} \qquad \& \qquad \vec{v}_n = T\,\vec{v}_{n-1}$$
$$= T^n \vec{u}_0 \qquad\qquad\qquad = T^n \vec{v}_0 .$$

Note that the points $\vec{v}_0, \vec{v}_1, \ldots$ bounce back and forth while converging to $\vec{0}$ because $|\varphi_2| < 1$ and $\varphi_2 < 0$.

For any other initial condition $\vec{f}_0$ the trajectory will be a mixture of these two kinds of trajectories. We can indicate this by drawing "flow lines" outside of the eigenlines as in the following picture



Now any trajectory will bounce back and forth between two of these flow lines (which have the shape of "hyperbolas"). I've drawn two example trajectories

The actual "Fibonacci numbers" are just the particular trajectory with initial condition

$$\vec{f}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Another Example

Consider the matrix $A = \begin{pmatrix} .8 & .3 \\ .2 & .7 \end{pmatrix}$.

Using a computer we find

$$A^2 = \begin{pmatrix} .70 & .45 \\ .30 & .55 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} .650 & .525 \\ .350 & .475 \end{pmatrix}$$

$$A^4 = \begin{pmatrix} .6250 & 0.5622 \\ .3750 & 0.4375 \end{pmatrix}$$

$$\vdots$$

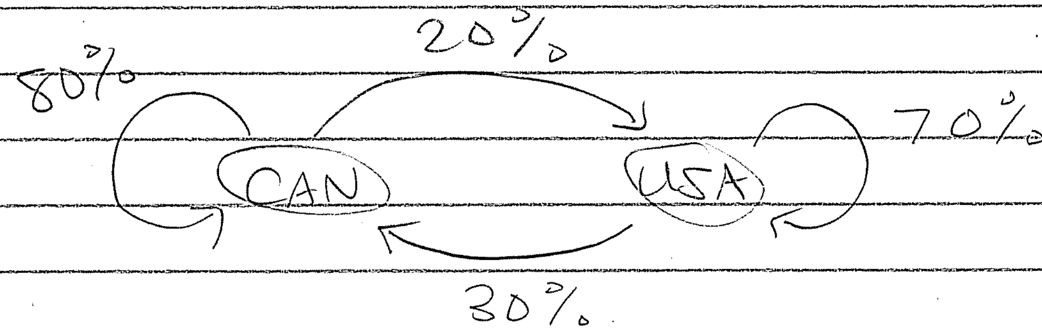
$$A^{10} = \begin{pmatrix} 0.600 & 0.599 \\ 0.399 & 0.401 \end{pmatrix}$$

It seems like

$$\lim_{n \to \infty} A^n = \begin{pmatrix} .6 & .6 \\ .4 & .4 \end{pmatrix}$$

What's going on here?

Consider a simple model. A species of bird lives in Canada and the USA. Every year there is a migration



Assume no birds are born or die.
In year n there are

c_n birds in CAN.
u_n birds in USA

How are $\begin{pmatrix} c_n \\ u_n \end{pmatrix}$ and $\begin{pmatrix} c_{n+1} \\ u_{n+1} \end{pmatrix}$ related?

Of the c_n birds in CAN now, $.8c_n$ stay and $.2c_n$ move. Of the u_n birds in USA now, $.7u_n$ stay and $.3u_n$ move.

Hence

$$c_{n+1} = .8c_n + .3u_n$$
$$u_{n+1} = .2u_n + .7u_n$$

i.e. $$\begin{pmatrix} c_{n+1} \\ u_{n+1} \end{pmatrix} = \begin{pmatrix} .8 & .3 \\ .2 & .7 \end{pmatrix} \begin{pmatrix} c_n \\ u_n \end{pmatrix}$$

$$\vec{v}_{n+1} = A\vec{v}_n$$

Say $\vec{v}_n$ is the state vector at time n
Say A is the transition matrix.

Example

Start with $\vec{v}_0 = \begin{pmatrix} 10 \\ 0 \end{pmatrix}$

Then $\vec{v}_1 = \begin{pmatrix} .8 & .3 \\ .2 & .7 \end{pmatrix} \begin{pmatrix} 10 \\ 0 \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \end{pmatrix}$

$$\vec{v}_2 = A\vec{v}_1 = \begin{pmatrix} .8 & .3 \\ .2 & .7 \end{pmatrix} \begin{pmatrix} 8 \\ 2 \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$$

$$\vec{v}_3 = A\vec{v}_2 = \begin{pmatrix} .8 & .3 \\ .2 & .7 \end{pmatrix} \begin{pmatrix} 7 \\ 3 \end{pmatrix} = \begin{pmatrix} 6.5 \\ 3.5 \end{pmatrix}$$

Q: 6.5 birds?
A: Yes. We're just dealing with <u>probabilities</u>.

In general we have

$$\begin{aligned} \vec{v}_n &= A\vec{v}_{n-1} \\ &= AA\vec{v}_{n-2} \\ &= AAA\vec{v}_{n-3} \\ &\ \vdots \\ &= \underbrace{AAA \cdots A}_{n \text{ times}} \vec{v}_0 \\ &= A^n \vec{v}_0 = \begin{pmatrix} .8 & .3 \\ .2 & .7 \end{pmatrix}^n \begin{pmatrix} 10 \\ 0 \end{pmatrix} \end{aligned}$$

Can we compute this?

Big Idea: We will say state $\vec{v}$ is an equilibrium of the system if

$$A\vec{v} = \vec{v} = 1\vec{v}.$$

An eigenvector with eigenvalue 1

If it exists, let's compute it.

Let $\vec{v} = \begin{pmatrix} c \\ u \end{pmatrix}$.

Then $A\vec{v} = \vec{v}$.

$$\Rightarrow \begin{pmatrix} .8 & .3 \\ .2 & .7 \end{pmatrix}\begin{pmatrix} c \\ u \end{pmatrix} = \begin{pmatrix} c \\ u \end{pmatrix}.$$

$$\Rightarrow \begin{aligned} .8c + .3u &= c \\ .2c + .7u &= u. \end{aligned}$$

$$\Rightarrow -.2c + .3u = 0$$
$\cancel{.2c - .3u = 0}$. redundant.

$$\Rightarrow \begin{aligned} .3u &= .2c \\ 3u &= 2c. \end{aligned}$$

$$\Rightarrow u/c = 2/3$$

The 1-eigenspace of A is the line

$$\begin{pmatrix} c \\ u \end{pmatrix} = t \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

In particular we have

$$A \begin{pmatrix} 6 \\ 4 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}.$$

6 birds in CAN
4 birds in USA is an equilibrium.

But we haven't yet explained why

$$A^n \begin{pmatrix} 10 \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 6 \\ 4 \end{pmatrix}$$

as $n \longrightarrow \infty$

To do this we need the <u>other</u> eigenvalue.

The characteristic equation of $\begin{pmatrix} .8 & .3 \\ .2 & .7 \end{pmatrix}$ is

$$(.8-\lambda)(.7-\lambda) - (.2)(.3) = 0$$

$$.56 - .8\lambda - .7\lambda + \lambda^2 - .06 = 0$$

$$\lambda^2 - 1.5\lambda + .5 = 0$$

$$2\lambda^2 - 3\lambda + 1 = 0$$

Hence the eigenvalues are

$$\lambda = \frac{3 \pm \sqrt{(-3)^2 - 4(1)(2)}}{2(2)} = \frac{3 \pm 1}{4}$$

$$= 1 \text{ or } .5$$

Let's compute the eigenvalues corresponding to eigenvalue .5

$$\begin{pmatrix} .8 & .3 \\ .2 & .7 \end{pmatrix} \begin{pmatrix} c \\ u \end{pmatrix} = .5 \begin{pmatrix} c \\ u \end{pmatrix}$$

$$\Rightarrow \quad .8c + .3u = .5c$$

$$.2c + .7u = .5u$$

$\Longrightarrow \quad .3c + .3u = 0$

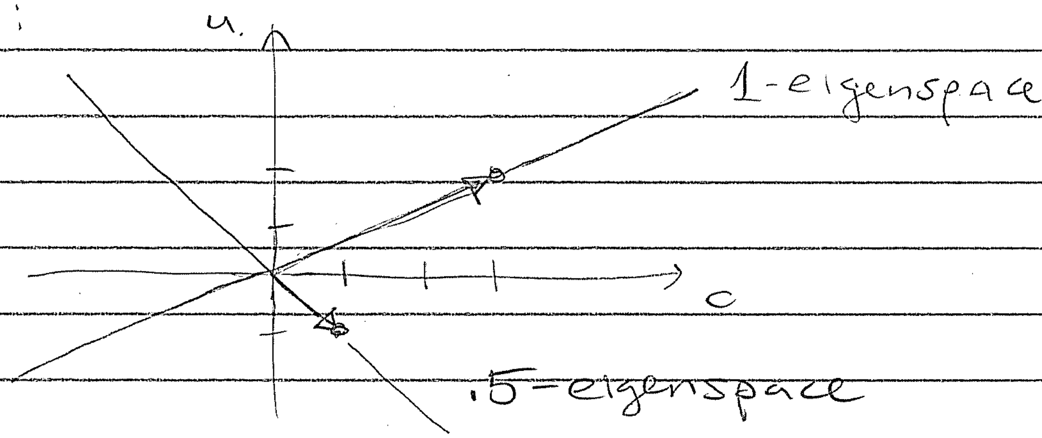
$\cancel{.2c + .2u = 0}$ Redundant ☺

$\Longrightarrow \quad c + u = 0.$

So the ".5-eigenspace" is the line

$$\begin{pmatrix} c \\ u \end{pmatrix} = t \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Picture:



Slogan: Once you know the eigenvectors, you should express everything in terms of them.

For example, let's express our initial state vector:

$$\begin{pmatrix} 10 \\ 0 \end{pmatrix} = 2\begin{pmatrix} 3 \\ 2 \end{pmatrix} + 4\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Then we have

$$A^n\begin{pmatrix} 10 \\ 0 \end{pmatrix} = A^n\left[2\begin{pmatrix} 3 \\ 2 \end{pmatrix} + 4\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right]$$

$$= 2A^n\begin{pmatrix} 3 \\ 2 \end{pmatrix} + 4A^n\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= 2\begin{pmatrix} 3 \\ 2 \end{pmatrix} + 4\left(\frac{1}{2}\right)^n\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 6 + 4/2^n \\ 4 - 4/2^n \end{pmatrix}$$

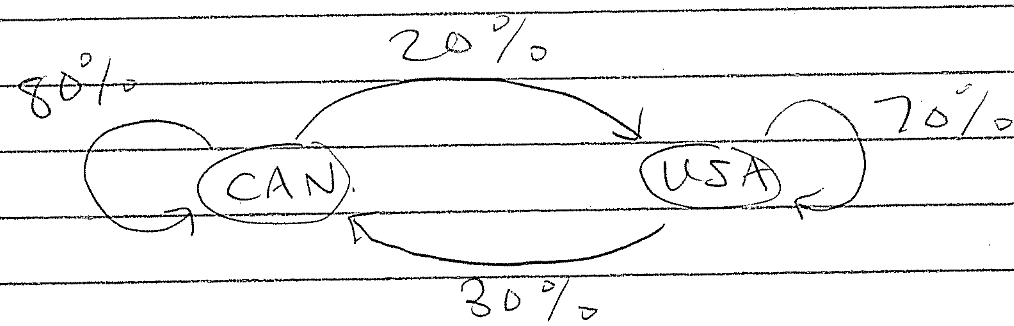
As $n \to \infty$ we have

$$A^n\begin{pmatrix} 10 \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 6+0 \\ 4+0 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$$

Phase Portraits

Today: Phase Portraits.

Recall the birds



and their transition matrix

$$A = \begin{pmatrix} .8 & .3 \\ .2 & .7 \end{pmatrix}$$

If we let c_n = # birds in CAN at year n

u_n = # birds in USA at year n.

Then we have.

$$\begin{pmatrix} c_n \\ u_n \end{pmatrix} = A \begin{pmatrix} c_{n-1} \\ u_{n-1} \end{pmatrix}$$

$$= A A \begin{pmatrix} c_{n-2} \\ u_{n-2} \end{pmatrix}$$

$$\vdots$$

$$= \underbrace{A A \cdots A}_{n \text{ times}} \begin{pmatrix} c_0 \\ u_0 \end{pmatrix}$$

$$= A^n \begin{pmatrix} c_0 \\ u_0 \end{pmatrix}$$

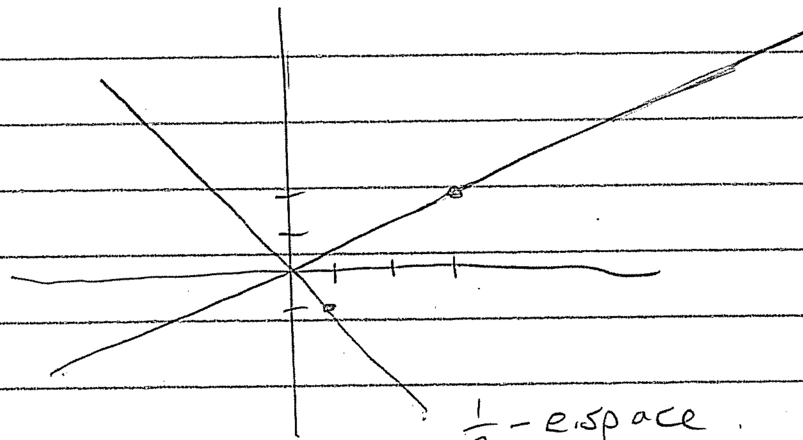
To solve this system, i.e., to find formulas for (c_n, u_n) in terms of (c_0, u_0), we must compute the eigenvalues/eigenvectors.

The e.values are 1 and .5.

The e.vectors are.

$$A t \begin{pmatrix} 3 \\ 2 \end{pmatrix} = 1 \cdot t \begin{pmatrix} 3 \\ 2 \end{pmatrix} \ \& \ A t \begin{pmatrix} 1 \\ -1 \end{pmatrix} = .5 \cdot t \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Picture:



This picture tells us a <u>lot</u>.
Suppose we start with $(c_0, u_0) = (10, 0)$.
This can be written in terms of eigenvectors as

$$\begin{pmatrix} 10 \\ 0 \end{pmatrix} = 2\begin{pmatrix} 3 \\ 2 \end{pmatrix} + 4\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 6 \\ 4 \end{pmatrix} + \begin{pmatrix} 4 \\ -4 \end{pmatrix}$$

Then we have

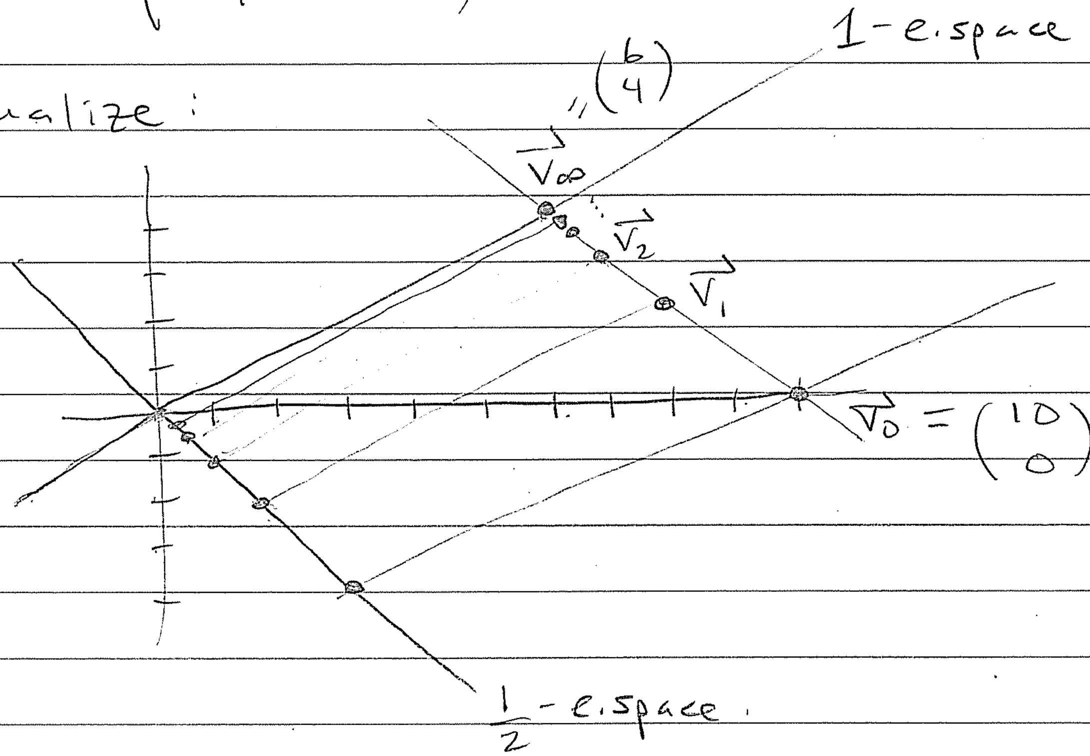
$$\begin{pmatrix} c_n \\ u_n \end{pmatrix} = A^n \begin{pmatrix} 10 \\ 0 \end{pmatrix} = A^n \left[\begin{pmatrix} 6 \\ 4 \end{pmatrix} + \begin{pmatrix} 4 \\ -4 \end{pmatrix} \right]$$

$$= A^n \begin{pmatrix} 6 \\ 4 \end{pmatrix} + A^n \begin{pmatrix} 4 \\ -4 \end{pmatrix}$$

$$= 1^n \begin{pmatrix} 6 \\ 4 \end{pmatrix} + \left(\frac{1}{2}\right)^n \begin{pmatrix} 4 \\ -4 \end{pmatrix}$$

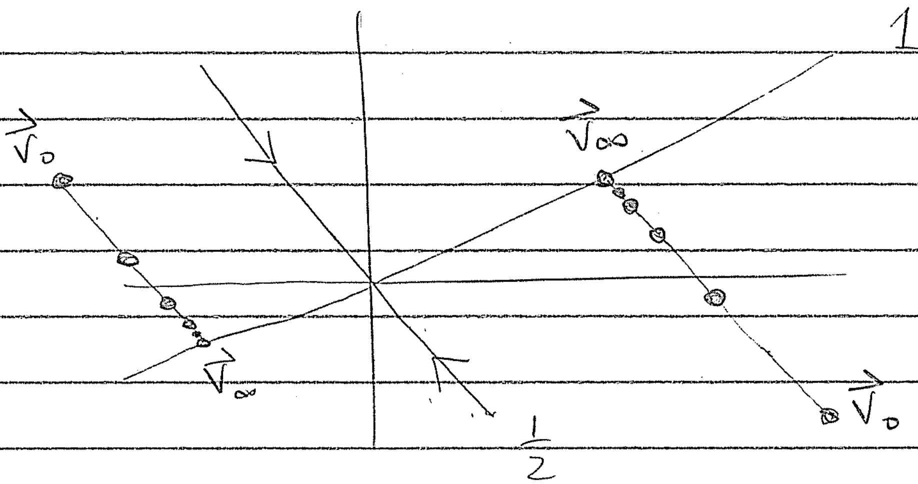
$$= \begin{pmatrix} 6 + 4/2^n \\ 4 - 4/2^n \end{pmatrix}$$

Visualize:



At each step, the state <u>halves</u> in the $(1,-1)$ direction and <u>stays</u> the <u>same</u> in the $(3,2)$ direction.

A general trajectory.



So the matrix A^∞ is a projection onto the line $t(3,2)$, but at a strange angle (i.e. not $90°$).

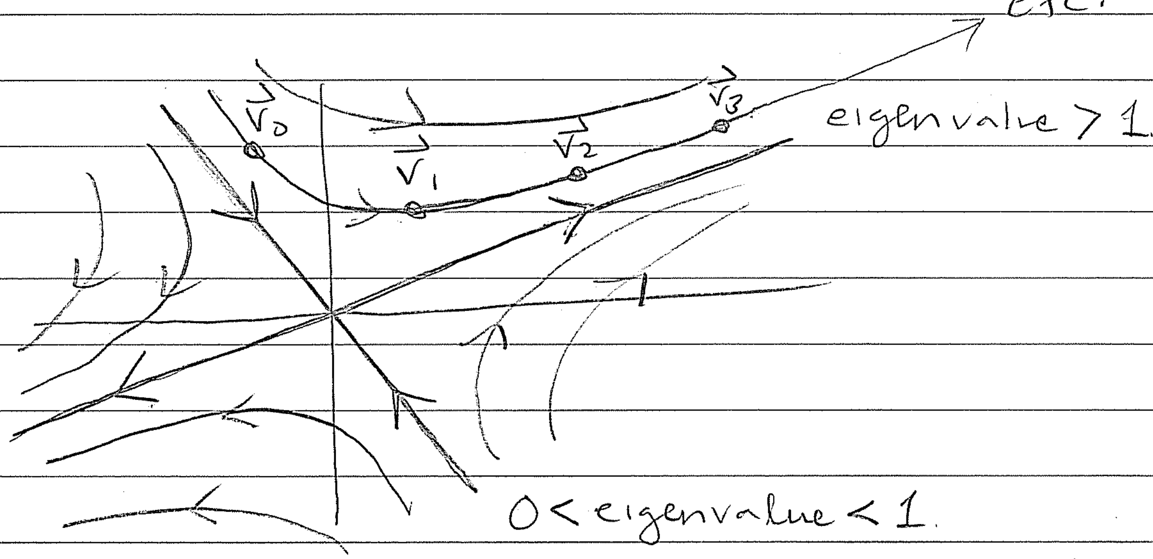
In fact, $A^\infty = \begin{pmatrix} .6 & .6 \\ .4 & .4 \end{pmatrix}$

The orthogonal projection would be

$$P = \frac{\begin{pmatrix} 3 \\ 2 \end{pmatrix} (3 \; 2)}{(3 \; 2) \begin{pmatrix} 3 \\ 2 \end{pmatrix}} = \frac{1}{13} \begin{pmatrix} 9 & 6 \\ 6 & 4 \end{pmatrix} \neq A^\infty.$$

Note: A^∞ has the same e.vectors but with e.values $1^\infty = \underline{1}$ and $(.5)^\infty = 0$

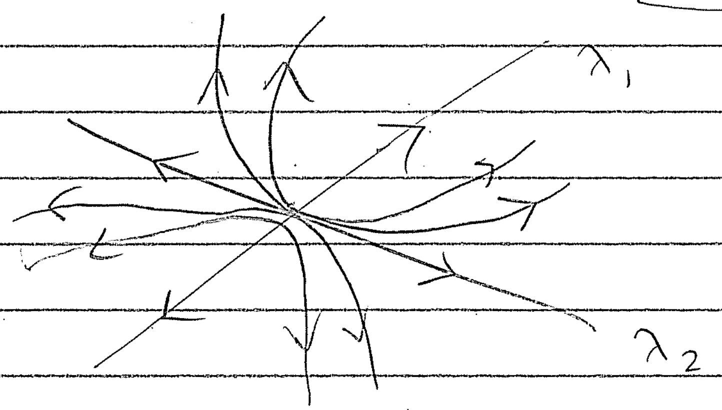
Q: What if we had a matrix with <u>this</u> eigen-information:



This is called a "phase portrait"
It shows us the <u>typical</u> <u>trajectories</u>.

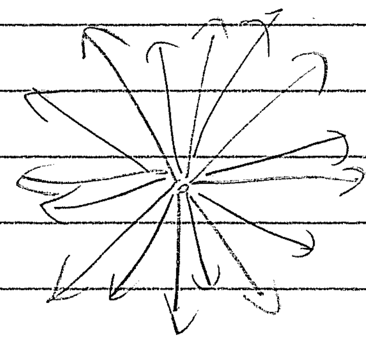
The evalues/evectors determine the behavior of the system.

Other Possibilities: $\boxed{\lambda_1 > \lambda_2 > 1}$

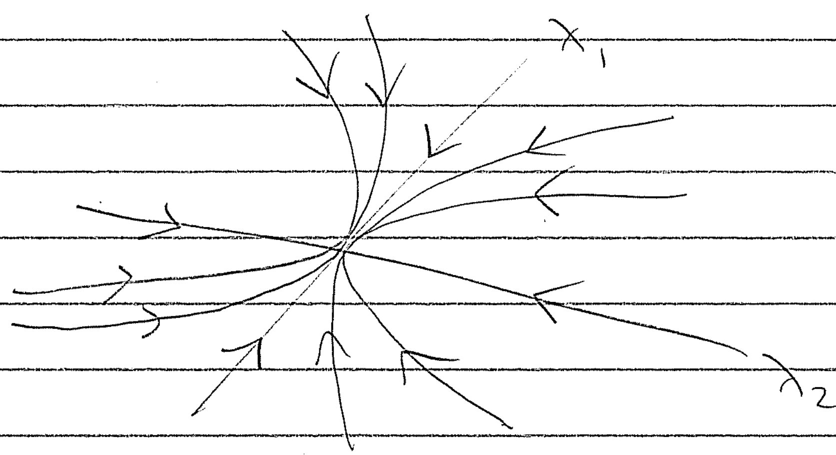


$\boxed{\lambda_1 = \lambda_2 > 1}$

Expands evenly in all directions



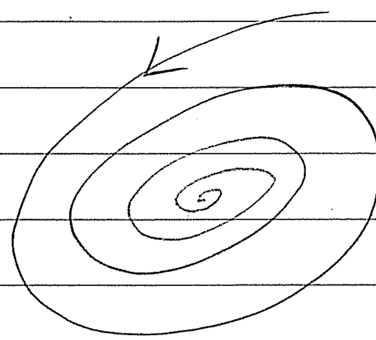
$\boxed{0 < \lambda_1 < \lambda_2 < 1}$



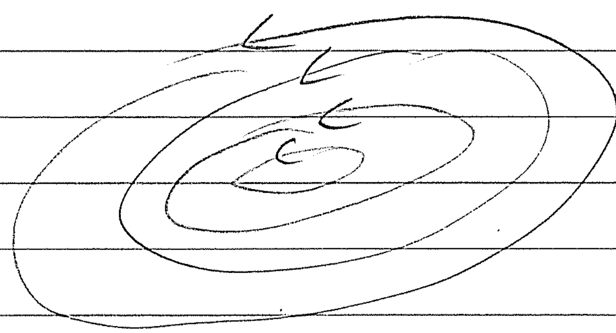
What if λ_1, λ_2 are complex?

Then the system will oscillate

$|\lambda_1| = |\lambda_2| < 1$

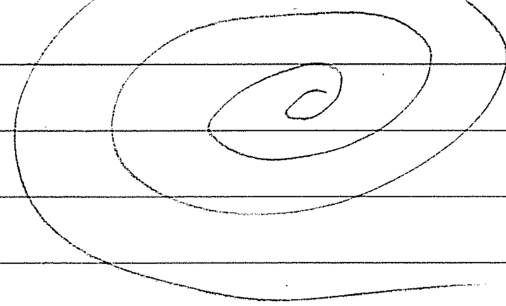


$|\lambda_1| = |\lambda_2| = 1$



closed orbits

$|\lambda_1| = |\lambda_2| > 1$



Final Example: Owls vs. Rats

$$O_{k+1} = (.5)\, O_k + (.4)\, R_k$$

$$R_{k+1} = -p\, O_k + (1.1)\, R_k.$$

Recurrence:

$$\begin{pmatrix} O_{k+1} \\ R_{k+1} \end{pmatrix} = \begin{pmatrix} .5 & .4 \\ -p & 1.1 \end{pmatrix} \begin{pmatrix} O_k \\ R_k \end{pmatrix}.$$

Initial Conditions NOT GIVEN.

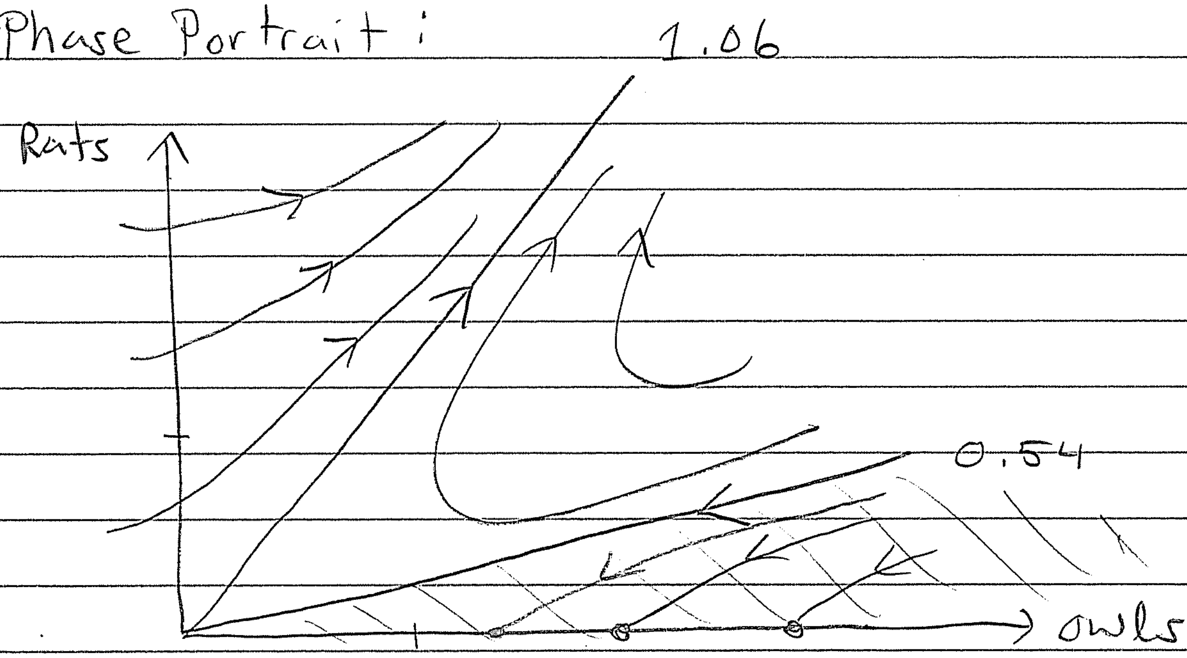
Draw the Phase Portrait for three values of p.

(1) $\underline{p = 0.056}$. We have

$$\begin{pmatrix} .5 & .4 \\ -0.056 & 1.1 \end{pmatrix} \begin{pmatrix} 1 \\ 1.4 \end{pmatrix} = (1.06) \begin{pmatrix} 1 \\ 1.4 \end{pmatrix}$$

and
$$\begin{pmatrix} .5 & .4 \\ -0.056 & 1.1 \end{pmatrix} \begin{pmatrix} 1 \\ .1 \end{pmatrix} = (0.54) \begin{pmatrix} 1 \\ .1 \end{pmatrix}$$

Phase Portrait:



If $O_0 > 10 R_0$ then everyone goes extinct.
Otherwise we have

$$\frac{R_n}{O_n} \to 1.4 \quad (40\% \text{ more rats than owls})$$

and both populations will grow
at $\approx 6\%$ per year.

I guess we'll call that "good".

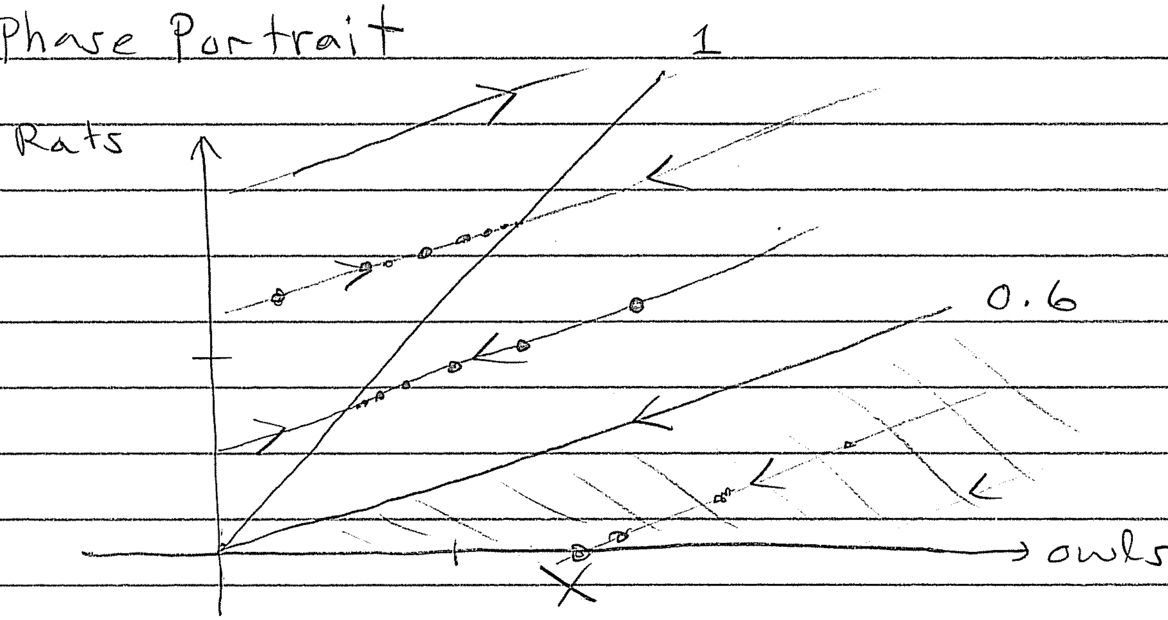
(2) $p = 0.125$ We have

$$\begin{pmatrix} .5 & .4 \\ -0.125 & 1.1 \end{pmatrix}\begin{pmatrix} 1 \\ 1.25 \end{pmatrix} = 1 \begin{pmatrix} 1 \\ 1.25 \end{pmatrix}$$

and

$$\begin{pmatrix} .5 & .4 \\ -0.125 & 1.1 \end{pmatrix}\begin{pmatrix} 1 \\ 0.25 \end{pmatrix} = (0.6) \begin{pmatrix} 1 \\ 0.25 \end{pmatrix}$$

Phase Portrait



If $O_0 > 4 R_0$ then everyone goes extinct.
Otherwise the populations approach a steady state with

$-0.25\, O_0 + R_0$ owls and
$-0.3125\, O_0 + 1.25\, R_0$ rats

We'll call this "okay".

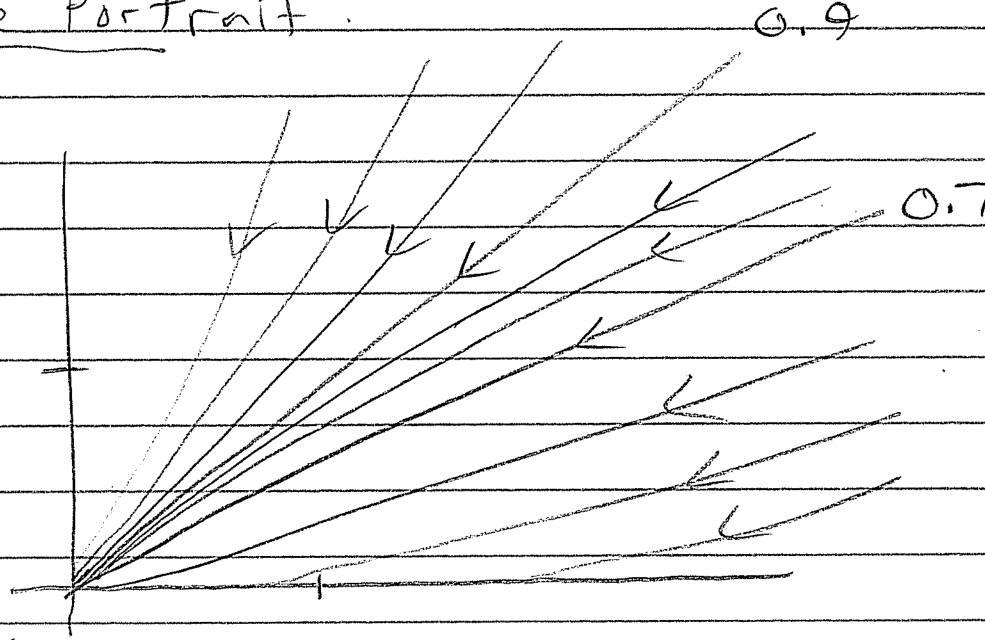
③ $\underline{p = 0.2}$. We have

$$\begin{pmatrix} .5 & .4 \\ -0.2 & 1.1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = (0.9) \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and

$$\begin{pmatrix} .5 & .4 \\ -0.2 & 1.1 \end{pmatrix} \begin{pmatrix} 1 \\ .5 \end{pmatrix} = (0.7) \begin{pmatrix} 1 \\ .5 \end{pmatrix}$$

<u>Phase Portrait</u>:



Everyone goes extinct no matter what O_0 and R_0 are

We'll call this "bad".

Memo to owls: Control your appetite.