Math 210 **Fall 2019**
Homework 5 **Drew Armstrong**

Problem 1. We say that P is a *projection matrix* if $P^T = P$ and $P^2 = P$.

(a) If P is a projection, show that $I - P$ is also a projection.
(b) Show that the projections P and $I - P$ satisfy $P(I - P) = 0$.
(c) Let A be any matrix of shape $m \times n$ so that $A^T A$ is square of shape $n \times n$. Assuming that the inverse $(A^T A)^{-1}$ exists, show that $P = A(A^T A)^{-1} A^T$ is a projection matrix. [We saw in class that this matrix projects onto the column space of A.]
(d) In the special case that A is a square and invertible, show that $P = A(A^T A)^{-1} A^T = I$. What does this mean?

Problem 2. Consider the plane $x + 2y + 2z = 0$ with normal vector $\mathbf{a} = (1, 2, 2)$.

(a) Use the formula from 1(c) to find the 3×3 matrix P that projects onto the line $t\mathbf{a}$. [Hint: Just let $A = \mathbf{a}$.]
(b) Use the matrix P to project the vector $\mathbf{b} = (1, -1, 1)$ onto the line.
(c) Find two vectors in the plane $x + 2y + 2z = 0$ and then use the formula from 1(c) to find the 3×3 matrix Q that projects onto the plane. [Hint: Let A be the 3×2 matrix whose columns are the two vectors that you found.]
(d) Use the matrix Q to project the vector $\mathbf{b} = (1, -1, 1)$ onto the plane.
(e) Finally, check that $P + Q = I$. Does this surprise you?

Problem 3. Shortcut. Let $\mathbf{a} = (1, 2, -1, 1)$ and consider the following hyperplane in $\mathbb{R}^4$:

$$\mathbf{a}^T \mathbf{x} = 1x_1 + 2x_2 - 1x_3 + 1x_4 = 0.$$

(a) Use 1(c) to compute the matrix P that projects onto the line $t\mathbf{a}$.
(b) We could also use 1(c) to compute the matrix Q that projects onto the hyperplane, but this would take too long. Instead, use the shortcut formula $Q = I - P$.
(c) Project the point $(1, 2, 3, 4)$ onto the hyperplane.

Problem 4. Find the best fit line $C + tD = b$ for the data points

$$\begin{pmatrix} t \\ b \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix},$$

using the following steps:

(a) Write down the matrix equation $A\mathbf{x} = \mathbf{b}$ that **would be** true if all four points were on the same line $C + tD = b$. This equation has no solution.
(b) Now write down the normal equation $A^T A \hat{\mathbf{x}} = A^T \mathbf{b}$ and solve it to find the least squares approximation $\hat{\mathbf{x}} = (C, D)$.
(c) Compute the error vector $\mathbf{e} = \mathbf{b} - A\hat{\mathbf{x}}$.
(d) Finally, draw the four data points along with their best fit line. Label the vertical errors with the entries of the error vector $\mathbf{e}$.

Problem 5. Find the best fit parabola $C + tD + Et^2 = b$ for the data points

$$\begin{pmatrix} t \\ b \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix},$$

using the following steps:

(a) Write down the matrix equation $A\mathbf{x} = \mathbf{b}$ that **would be** true if all four points were on the same parabola $C + tD + t^2E = b$. This equation has no solution.
(b) Now write down the normal equation $A^T A\hat{\mathbf{x}} = A^T\mathbf{b}$ and solve it to find the least squares approximation $\hat{\mathbf{x}} = (C, D, E)$.
(c) Compute the error vector $\mathbf{e} = \mathbf{b} - A\hat{\mathbf{x}}$.
(d) Finally, draw the four data points along with their best fit parabola. Label the vertical errors with the entries of the error vector $\mathbf{e}$.