Math 210 **Fall 2019**
Homework 4 **Drew Armstrong**

Problem 1. Consider the following three matrices:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Compute the following matrix products or explain why they do not exist:

$$AB, \quad BA, \quad ABC, \quad CAB, \quad C^TAB.$$

Answer: The matrix CAB does not exist. The others are given by

$$AB = \begin{pmatrix} 2 & 0 \\ 4 & -1 \end{pmatrix}, \quad BA = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & -1 & -2 \end{pmatrix}, \quad ABC = \begin{pmatrix} 4 \\ 7 \end{pmatrix}, \quad C^TAB = \begin{pmatrix} 8 & -1 \end{pmatrix}.$$

Problem 2.

(a) Let A be an $m \times n$ matrix and let $\mathbf{e}_j \in \mathbb{R}^n$ be the standard basis vector with 1 in the jth position and 0 in every other position. Explain why

$$A\mathbf{e}_j = (j\text{th column of } A).$$

(b) Use part (a) to find the 2×2 matrix R that rotates every vector in $\mathbb{R}^2$ counterclockwise by 45°. [Hint: What does R do the basis vectors $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$?]

(c) Use part (b) to rotate the vector $(1, 3)$ counterclockwise by 45°.

(a) We can take it as a definition that (jth column of AB)= A(jth column of B). In the special case that $B = I$ is the identity matrix we get

$$(j\text{th column of A}) = (j\text{th column of AI}) = A(j\text{th column of I}) = A\mathbf{e}_j.$$

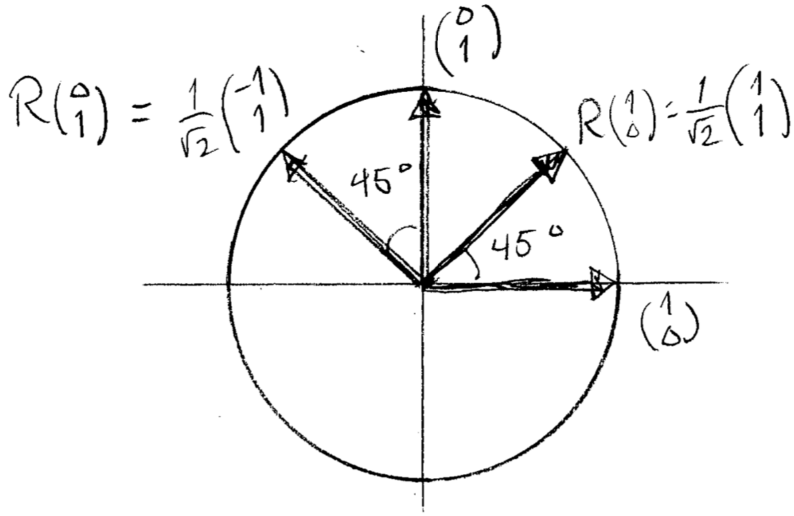
Or we can use the definition that

$$A\mathbf{x} = A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \sum_{j=1}^{n} x_j (j\text{th column of A}).$$

Since $\mathbf{e}_j$ has coordinates $e_{jj} = 1$ and $e_{ji} = 0$ if $i \neq j$ then we obtain

$$A\mathbf{e}_j = \sum_{i=1}^{n} e_{ji}(j\text{th column of } A) = 1(j\text{th column of } A) + 0(\text{all the other columns}).$$

(b) The following picture shows that $R(1, 0) = (1/\sqrt{2}, 1/\sqrt{2})$ and $R(0, 1) = (-1/\sqrt{2}, 1/\sqrt{2})$:



It follows from part (a) that

$$R = \left(R\begin{pmatrix}1\\0\end{pmatrix} \quad R\begin{pmatrix}0\\1\end{pmatrix} \right) = \begin{pmatrix}1/\sqrt{2} & -1/\sqrt{2}\\ 1/\sqrt{2} & 1/\sqrt{2}\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}.$$

(c) Then it follows from part (b) that

$$R\begin{pmatrix}1\\3\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}1\\3\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}-2\\4\end{pmatrix}.$$

Try doing that without linear algebra.

Problem 3. In general, I claim that the following 2×2 matrix rotates every vector counter-clockwise by angle θ:

$$R_\theta = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}.$$

(a) Find the matrices R_{0°, R_{30°, R_{60° and R_{90°.
(b) Compute the matrix product $R_{30^\circ} \cdot R_{60^\circ}$.
(c) Give a geometric reason to explain why $R_\alpha R_\beta = R_{\alpha+\beta}$ for all angles α and β.
(d) Use the result of part (c) to prove the trigonometric angle sum identities:

$$\begin{cases} \cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta\\ \sin(\alpha+\beta) &= \cos\alpha\sin\beta + \sin\alpha\cos\beta.\end{cases}$$

(a) Answer:

$$R_{0^\circ} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \quad R_{30^\circ} = \frac{1}{2}\begin{pmatrix}\sqrt{3} & -1\\ 1 & \sqrt{3}\end{pmatrix}, \quad R_{60^\circ} = \frac{1}{2}\begin{pmatrix}1 & -\sqrt{3}\\ \sqrt{3} & 1\end{pmatrix}, \quad R_{90^\circ} = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}.$$

(b) Rotating by 60° and then by 30° is the same as rotating by 90°:

$$R_{30^\circ} \cdot R_{60^\circ} = \frac{1}{2}\begin{pmatrix}\sqrt{3} & -1\\ 1 & \sqrt{3}\end{pmatrix}\frac{1}{2}\begin{pmatrix}1 & -\sqrt{3}\\ \sqrt{3} & 1\end{pmatrix} = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} = R_{90^\circ}.$$

(c) More generally, rotating by β and then by α (or the other way around) is the same as rotating by $\alpha+\beta$:

$$R_\alpha R_\beta = R_\beta R_\alpha = R_{\alpha+\beta}.$$

There is not much more to say about this.

(d) It follows from (c) that for any real numbers α and β we have

$$\begin{aligned} R_{\alpha+\beta} &= R_\alpha R_\beta \\ \begin{pmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{pmatrix} &= \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} \\ \begin{pmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{pmatrix} &= \begin{pmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -\cos\alpha\sin\beta - \sin\alpha\cos\beta \\ \cos\alpha\sin\beta + \sin\alpha\cos\beta & \cos\alpha\cos\beta - \sin\alpha\sin\beta \end{pmatrix} \end{aligned}$$

Comparing entries gives the angle sum identities. [Remark: These identities should never be memorized because they follow immediately from (c). Better to memorize the entries of the rotation matrix R_θ.]

Problem 4. Let A be a matrix of shape 2×3 and assume for contradiction that there exists an inverse matrix B of shape 3×2 such that

$$AB = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad BA = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(a) Explain why the linear system $A\mathbf{x} = \mathbf{0}$ has at least one nonzero solution $\mathbf{x} \neq \mathbf{0}$. [Hint: Consider the RREF.]
(b) It follows from (a) that $(BA)\mathbf{x} = B(A\mathbf{x}) = B\mathbf{0} = \mathbf{0}$ for some nonzero vector $\mathbf{x} \neq \mathbf{0}$. Explain why this is a contradiction.

[Remark: The same argument shows that any invertible matrix must be **square**.]

(a) Geometrically, the system $A\mathbf{x} = \mathbf{0}$ represents the intersection of two planes through the origin in $\mathbb{R}^3$. These planes must intersect in a line or a full plane, hence there must exist a nonzero solution (in fact, infinitely many). Alternatively, the RREF of the system $A\mathbf{x} = \mathbf{0}$ must have a non-pivot column because there can only be one pivot per row. Hence the system has infinitely many solutions. [The same argument works for any matrix with more columns than rows.]

(b) Now we know that $A\mathbf{0} = \mathbf{0}$ and $A\mathbf{x} = \mathbf{0}$ for some $\mathbf{x} \neq \mathbf{0}$. Since the function A sends the two different points $\mathbf{x}$ and $\mathbf{0}$ to the same point $\mathbf{0}$ we know that it cannot be inverted.[1] To see this directly, assume for contradiction that there exists some matrix with $BA = I$. Then we must have

$$\mathbf{x} = I\mathbf{x} = (BA)\mathbf{x} = B(A\mathbf{x}) = B\mathbf{0} = \mathbf{0},$$

which contradicts the fact that $\mathbf{x} \neq \mathbf{0}$. [The same argument shows every invertible matrix must be square.]

[1] Technically speaking: We say that a function $f : S \to T$ is invertible if (1) it is *injective*, meaning that $f(x) = f(y)$ implies $x = y$ and (2) it is *surjective*, meaning that for every $t \in T$ there exists some $s \in S$ such that $f(s) = t$. We have just shown that the matrix function A is not injective, hence it cannot be invertible.

Problem 5. Let A be some matrix and suppose that we have

$$A\mathbf{b}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad A\mathbf{b}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad A\mathbf{b}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

for some vectors $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$. Now let $B = \begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{pmatrix}$ be the matrix with columns $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$. Compute the matrix product AB.

Recall that (jth column of AB) $= A$(jth column of B) $= A\mathbf{b}_j$. It follows that

$$AB = \begin{pmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & A\mathbf{b}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I.$$

In other words, $B = A^{-1}$.

Problem 6. Not every square matrix is invertible. Consider the following matrix:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & c \end{pmatrix} \qquad \text{for some constant } c.$$

(a) If $c = 0$, find some specific nonzero vector $\mathbf{x} \neq \mathbf{0}$ such that $A\mathbf{x} = \mathbf{0}$. In this case it follows as in Problem 4 that A is **not invertible**.

(b) If $c \neq 0$ then the matrix A is invertible. Compute the RREF of the augmented matrix $(A|I)$ to find the inverse. [Remark: This method works because of Problem 5. You are solving three linear systems simultaneously to find the column vectors $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ of the inverse matrix.]

(a) For any real number t we observe that

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

By choosing any $t \neq 0$ we conclude that the matrix on the left is not invertible. [Recall: If a matrix sends any nonzero vector to zero then it is not invertible.]

(b) On the other hand, if $c \neq 0$ then we can compute the inverse:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & c & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & c & -1 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & c & 0 & -1 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1/c & 1/c \end{array}\right)$$

It follows that

$$A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & c \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1/c & 1/c \end{pmatrix} = B.$$

You should verify this by computing the products AB and BA.