Math 210 **Fall 2019**
Homework 4 **Drew Armstrong**

Problem 1. Consider the following three matrices:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Compute the following matrix products or explain why they do not exist:

$$AB, \quad BA, \quad ABC, \quad CAB, \quad C^T AB.$$

Problem 2.

(a) Let A be an $m \times n$ matrix and let $\mathbf{e}_j \in \mathbb{R}^m$ be the standard basis vector with 1 in the jth position and 0 in every other position. Explain why

$$A\mathbf{e}_j = (j\text{th column of } A).$$

(b) Use part (a) to find the 2×2 matrix R that rotates every vector in $\mathbb{R}^2$ counterclockwise by $45°$. [Hint: What does R do the basis vectors $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$?]

(c) Use part (b) to rotate the vector $(1, 3)$ counterclockwise by $45°$.

Problem 3. In general, I claim that the following 2×2 matrix rotates every vector counterclockwise by angle θ:

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

(a) Find the matrices $R_{0°}$, $R_{30°}$, $R_{60°}$ and $R_{90°}$.

(b) Compute the matrix product $R_{30°} \cdot R_{60°}$.

(c) Give a geometric reason to explain why $R_\alpha R_\beta = R_{\alpha+\beta}$ for all angles α and β.

(d) Use the result of part (c) to prove the trigonometric angle sum identities:

$$\begin{cases} \cos(\alpha + \beta) & = & \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \sin(\alpha + \beta) & = & \cos\alpha\sin\beta + \sin\alpha\cos\beta. \end{cases}$$

Problem 4. Let A be a matrix of shape 2×3 and assume for contradiction that there exists an inverse matrix B of shape 3×2 such that

$$AB = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad BA = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(a) Explain why the linear system $A\mathbf{x} = \mathbf{0}$ has at least one nonzero solution $\mathbf{x} \neq \mathbf{0}$. [Hint: Consider the RREF.]

(b) It follows from (a) that $(BA)\mathbf{x} = B(A\mathbf{x}) = B\mathbf{0} = \mathbf{0}$ for some nonzero vector $\mathbf{x} \neq \mathbf{0}$. Explain why this is a contradiction.

[Remark: The same argument shows that any invertible matrix must be **square**.]

Problem 5. Let A be some matrix and suppose that we have

$$A\mathbf{b}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad A\mathbf{b}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad A\mathbf{b}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

for some vectors $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$. Now let $B = \begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{pmatrix}$ be the matrix with columns $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$. Compute the matrix product AB.

Problem 6. Not every square matrix is invertible. Consider the following matrix:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & c \end{pmatrix} \qquad \text{for some constant } c.$$

(a) If $c = 0$, find some specific nonzero vector $\mathbf{x} \neq \mathbf{0}$ such that $A\mathbf{x} = \mathbf{0}$. In this case it follows as in Problem 4 that A is **not invertible**.

(b) If $c \neq 0$ then the matrix A is invertible. Compute the RREF of the augmented matrix $(A|I)$ to find the inverse. [Remark: This method works because of Problem 5. You are solving three linear systems simultaneously to find the column vectors $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ of the inverse matrix.]