Math 210 F **Drew Armstrong**
Final Exam A **Wed, Apr 27, 2016**

This is a closed book test. No electronic devices are allowed. If two students submit exams in which any solution has been copied, **both students will receive a score of zero**. There are 6 problems and 7 pages. Each page is worth 6 points, for a total of 42 points.

Problem 1. Consider the points $\vec{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\vec{y} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$ in the Cartesian plane.

(a) Draw the collection of points $s\vec{x} + t\vec{y}$ where $0 \leq s \leq 1$ and $0 \leq t \leq 1$.

(b) Draw the collection of points $t\vec{x} + t\vec{y}$ for all t.

(c) Draw the collection of points $t\vec{x} + (1-t)\vec{y}$ for all t.

Problem 2. Consider the same vectors $\vec{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\vec{y} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$ from Problem 1.

(a) Compute the lengths of $\vec{x}$ and $\vec{y}$.

(b) Compute the cosine of the angle between $\vec{x}$ and $\vec{y}$.

(c) Compute the otrhogonal projection of the point $\vec{x}$ onto the line $t\vec{y}$.

Problem 3. Consider the vectors $\vec{u} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$ in $\mathbb{R}^3$.

(a) Find a vector in $\mathbb{R}^3$ that is perpendicular to **both** of the vectors $\vec{u}$ and $\vec{v}$.

(b) Use your answer from part (a) to find the equation of the plane $s\vec{u} + t\vec{v}$.

(c) Compute the 3×3 matrix that projects orthogonally onto the plane from part (b).

Problem 4. Consider the following system of 3 linear equations in 4 unknowns:

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 1 \\ 2x_1 + 2x_2 + 3x_3 + 4x_4 = 2 \\ 3x_1 + 3x_2 + 4x_3 + 5x_4 = 3 \end{cases}$$

(a) Put the system in Reduced Row Echelon Form (RREF).

(b) Tell me the pivot and non-pivot variables.

(c) Write down the complete solution of the system.

Problem 5. Consider the matrix $T = \begin{pmatrix} 0.4 & 0.8 \\ 0.6 & 0.2 \end{pmatrix}$.

(a) Write down the characteristic equation of the matrix T. I'll just tell you that its two roots are 1 and -0.4 (you don't have to check this).

(b) Find an eigenvector of T corresponding to eigenvalue $\lambda = 1$.

(c) Find an eigenvector of T corresponding to eigenvalue $\lambda = -0.4$.

(d) Express $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ as a linear combination of the eigenvectors from parts (b) and (c).

(e) Finally, consider the linear recurrence relation $\vec{v}_n = T\vec{v}_{n-1}$ with initial condition $\vec{v}_0 = (3, 4)$. Use all of your previous work to find a "closed form" solution for the n-th vector $\vec{v}_n$.

Problem 6. Let A be a square matrix.

(a) If A is invertible, explain why $\lambda = 0$ **cannot** be an eigenvalue of A. [Hint: Suppose that we have $A\vec{x} = 0\vec{x} = \vec{0}$ for some vector $\vec{x} \neq \vec{0}$. Then ...]

(b) If A is invertible and λ is an eigenvalue of A, explain why λ^{-1} is an eigenvalue of the inverse matrix A^{-1}. [Hint: Suppose that we have $A\vec{x} = \lambda\vec{x}$ for some vector $\vec{x} \neq \vec{0}$. Then ...]

(c) Suppose that A is a 2×2 matrix with eigenvalues $\lambda = 3$ and $\lambda = 4$. In this case, tell me the eigenvalues of the matrix A^n.

Math 210 E **Drew Armstrong**
Final Exam **Monday, May 6, 2013**

This is a closed book test. No electronic devices are allowed. If two students submit exams in which any solution has been copied, **both students will receive a score of zero**. There are 7 pages and 7 problems.

Problem 1. Consider the plane $x + 2y - z = 0$ in $\mathbb{R}^3$.

(a) Tell me a normal vector to the plane.

(b) Tell me a normal vector of length 1.

(c) Compute the matrix Q that projects orthogonally onto the normal line.

(d) Compute the matrix P that projects orthogonally onto the plane. [Hint: Use your answer from part (c) to save time.]

Problem 2. The following matrix **rotates** vectors in $\mathbb{R}^2$ **counterclockwise** by 53.13°:

$$R = \frac{1}{5}\begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix}.$$

(You can just believe this. You don't have to show it.)

(a) Rotate the column vector $(1, 1)$ **counterclockwise** by 53.13°.

(b) Compute the matrix that rotates vectors **clockwise** by 53.13°.

(c) Rotate the column vector $(1, 1)$ **clockwise** by 53.13°.

(d) Compute the matrix that rotates **counterclockwise** by $106.26°(= 2 \times 53.13°)$.

Problem 3. Consider the following three planes in $\mathbb{R}^3$:

$$x + y + z = 0, \tag{1}$$
$$x + 2y - z = 0, \tag{2}$$
$$x + 0y + z = 1. \tag{3}$$

(a) Compute the intersection of the **first and second planes**. [Hint: It's a line.]

(b) Compute the intersection of **second and third planes**. [Hint: It's a line.]

(c) Compute the intersection of the lines from parts (a) and (b). [Hint: It's a point.]

Problem 4. Consider the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$.

(a) Use the Gauss-Jordan method to compute the inverse of A.

(b) Solve the system $A\vec{x} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$. [Hint: Use your answer from (a) to save time.]

Problem 5. Consider the following system:

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & b \\ 0 & 1 & 1 \end{pmatrix} \vec{x} = \begin{pmatrix} 1 \\ 0 \\ c \end{pmatrix}.$$

Tell me some values for b and c such that

(a) the system has a **unique solution** $\vec{x}$.

(b) the system has **no solution** $\vec{x}$.

(c) the system has **infinitely many solutions** $\vec{x}$.

Problem 6. Consider the matrix $A = \begin{pmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_m \end{pmatrix}$, where $\vec{a}_i$ is the ith **column**. Suppose that A has n rows, so the columns are vectors in $\mathbb{R}^n$.

(a) Let $\vec{x}$ be a vector in $\mathbb{R}^n$. Write a single matrix equation to say that $\vec{x}$ is perpendicular to all the columns of A.

(b) You can think of your matrix equation from part (a) as a system of **how many** linear equations, in **how many** unknowns?

(c) The solution to your equation in part (a) most likely has **how many dimensions**?

(d) Now let $\vec{b}$ be any point in $\mathbb{R}^n$ and let $\vec{p}$ be the point in the column space of A that is **closest** to $\vec{b}$. Write a formula for $\vec{p}$ in terms of A and $\vec{b}$.

Problem 7. Let $\vec{u}$ and $\vec{v}$ be vectors in $\mathbb{R}^n$ with $\vec{u}^T\vec{v} \neq 0$, and consider the $n \times n$ matrix

$$A = \frac{\vec{u}\,\vec{v}^T}{\vec{u}^T\vec{v}}.$$

(a) For all vectors $\vec{x}$, show that $A\vec{x}$ is on the line generated by $\vec{u}$.

(b) The line generated by $\vec{u}$ is an eigenspace for A. What is the eigenvalue?

(c) Now let $\vec{x}$ be any vector **perpendicular** to $\vec{v}$. Show that $A\vec{x} = \vec{0}$.

(d) Is the matrix A invertible? If so, tell me its inverse.