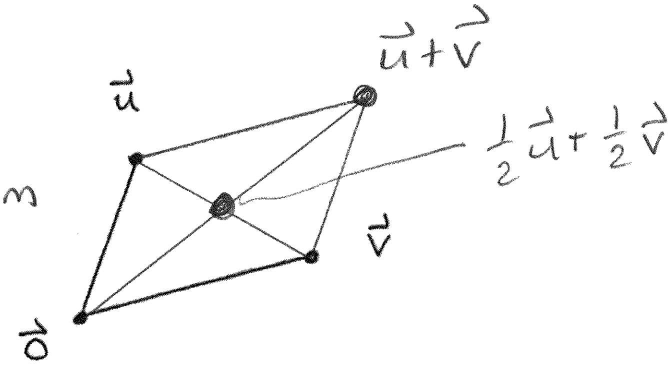
Intro to Linear Algebra
Quiz 1

Summer 2017
Drew Armstrong

Problem 1. Consider two points $\vec{u}$ and $\vec{v}$ in the Cartesian plane.

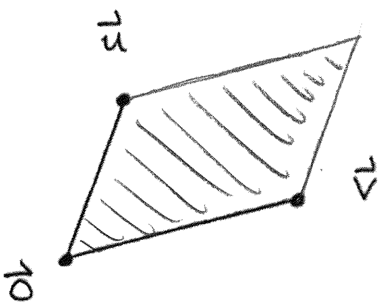
(a) Draw the points $\vec{u}+\vec{v}$ and $\frac{1}{2}\vec{u}+\frac{1}{2}\vec{v}$:

vertex and midpoint of the parallelogram



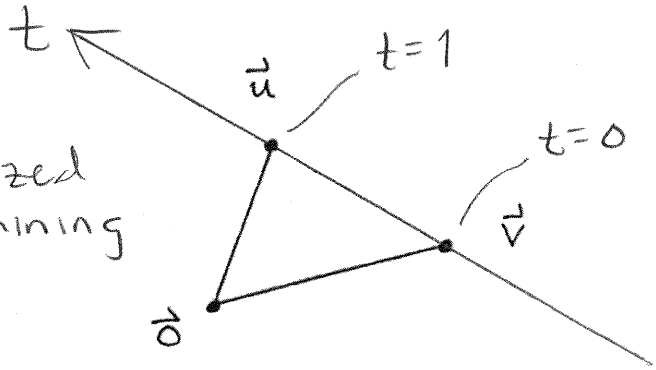
(b) Draw the set of all points $a\vec{u}+b\vec{v}$ where $0 \leqslant a \leqslant 1$ and $0 \leqslant b \leqslant 1$:

filled parallelogram



(c) Draw the set of all points $t\vec{u}+(1-t)\vec{v}$ where t ranges over all numbers:

parametrized line containing $\vec{u}$ & $\vec{v}$



Problem 2. Consider two vectors $\vec{v}$ and $\vec{w}$ with the following properties:

$$\vec{v} \bullet \vec{v} = \|v\|^2 = 1, \qquad \vec{w} \bullet \vec{w} = \|w\|^2 = 2 \qquad \text{and} \qquad \vec{v} \bullet \vec{w} = 0.$$

(a) Compute the **length** of the vector $\vec{v} + \vec{w}$.

$$\begin{aligned} \|\vec{v}+\vec{w}\|^2 &= (\vec{v}+\vec{w}) \circ (\vec{v}+\vec{w}) \\ &= \vec{v} \circ \vec{v} + 2\vec{v} \circ \vec{w} + \vec{w} \circ \vec{w} \\ &= 1 + 2(0) + 2 = 3 \end{aligned}$$

$$\Longrightarrow \quad \|\vec{v}+\vec{w}\| = \sqrt{3}$$

(b) Compute the **length** of the vector $\vec{v} - 2\vec{w}$.

$$\begin{aligned} \|\vec{v}-2\vec{w}\|^2 &= (\vec{v}-2\vec{w}) \circ (\vec{v}-2\vec{w}) \\ &= \vec{v} \circ \vec{v} - 4\vec{v} \circ \vec{w} + 4\vec{w} \circ \vec{w} \\ &= 1 - 4(0) + 4(2) = 9 \end{aligned}$$

$$\Longrightarrow \quad \|\vec{v}-2\vec{w}\| = \sqrt{9} = 3.$$

(c) Compute the **cosine of the angle** between the vectors $\vec{v} + \vec{w}$ and $\vec{v} - 2\vec{w}$.

First: $(\vec{v}+\vec{w}) \circ (\vec{v}-2\vec{w}) = \vec{v} \circ \vec{v} - \vec{v} \circ \vec{w} - 2\vec{w} \circ \vec{w}$

$$= 1 - 0 - 2(2) = -3.$$

Then:

$$\cos\theta = \frac{(\vec{v}+\vec{w}) \circ (\vec{v}-2\vec{w})}{\|\vec{v}+\vec{w}\| \cdot \|\vec{v}-2\vec{w}\|} = \frac{-3}{\sqrt{3} \cdot 3} = \frac{-1}{\sqrt{3}}.$$

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Quiz 2

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Problem 1.

(a) Find the equation of the line that is perpendicular to the vector $(1, 2)$ and contains the point $(0, 0)$.

$$\begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 0$$

$$\boxed{x + 2y = 0}$$

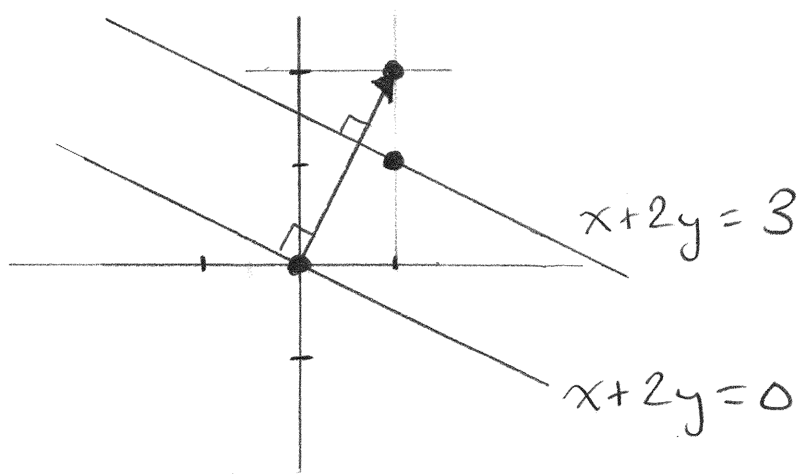
(b) Find the equation of the line that is perpendicular to the vector $(1, 2)$ and contains the point $(1, 1)$.

Eqn is $x + 2y = c$ for some c.
Plug in the point $(1,1)$:

$$1 + 2(1) = c$$
$$3 = c$$

So the equation is $\boxed{x + 2y = 3}$

(c) Draw the lines from parts (a) and (b) on the same pair of axes. Label each line by its equation.



Problem 2.

(a) Find a parametrization for the line in 3D that contains the point $(1, 0, 0)$ and is parallel to the vector $(1, 2, 3)$.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

(b) Compute the intersection of the line from part (a) with the plane $x - y + z = 5$.

Substitute $x = 1+t$, $y = 2t$, $z = 3t$ to get

$$(1+t) - (2t) + (3t) = 5$$
$$1 + 2t = 5$$
$$2t = 4$$
$$t = 2$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix}$$

(c) Is the line from part (a) perpendicular to the plane from part (b)? Why or why not?

NO. Because the direction vector $(1,2,3)$ is not parallel to the perpendicular vector $(1,-1,1)$ of the plane.

$$(1,2,3) \neq r(1,-1,1)$$

for any r

Intro to Linear Algebra
Quiz 3

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Problem 1. Consider the following system of linear equations:

$$\begin{cases} x + 2y + 0 = -1 \\ x + 2y + z = 0 \\ x + 2y + 2z = 1 \end{cases}$$

(a) Put the system in reduced row echelon form (RREF).

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & -1 \\ 1 & 2 & 1 & 0 \\ 1 & 2 & 2 & 1 \end{array}\right) \Longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 2 \end{array}\right)$$

$$\Longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \begin{cases} x + 2y + 0 = -1 \\ 0 + 0 + z = 1 \\ 0 + 0 + 0 = 0 \end{cases}$$

(b) Use your answer from part (a) to write down the complete solution.

Let $y = t$ be free. Then

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1-2t \\ t \\ 1 \end{pmatrix} = \underbrace{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}}.$$

This is the line containing the point $(-1,0,1)$ and parallel to the vector $(-2,1,0)$.

Problem 2. Consider the same system of equations again:

$$\begin{cases} x + 2y + 0 = -1 \\ x + 2y + z = 0 \\ x + 2y + 2z = 1 \end{cases}$$

1
2
3

(a) The three linear equations represent three planes living in 3D. Tell me three vectors that are perpendicular to these three planes.

Plane 1 is $\perp$ to $(1,2,0)$
plane 2 is $\perp$ to $(1,2,1)$
plane 3 is $\perp$ to $(1,2,2)$.

(b) Fill in the blanks. Let E_1, E_2, E_3 represent the three linear equations. The reason that the solution is a line (instead of a point) is because there exists a non-trivial relation among the equations:

$$E_3 = \underline{-1} \cdot E_1 + \underline{2} \cdot E_2$$

(c) Fill in the blanks. The equation from part (b) has the following consequences:

If the point (x, y, z) satisfies the first and second equations then it also

satisfies the third equation.

Geometrically, the intersection of the first and second planes is contained in

the third plane.