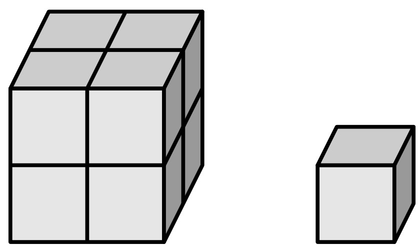
Surface Area of a Sphere

Our goal is to prove that a sphere of radius r has surface area $4\pi r^2$. We are looking for a convincing geometric proof that does **not** use the Fundamental Theorem of Calculus.

• If the dimensions of a 3D shape are multplied by 1/2, what happens to the volume?

First of all, what happens to a cube? If a cube with side length ℓ has volume ℓ^3 then a cube with side length $\ell/2$ has volume $(\ell/2)^3 = \ell^3/8$. Thus the volume is multiplied by 1/8. This is easy to see because the original cube can be divided into 8 of the smaller cubes.

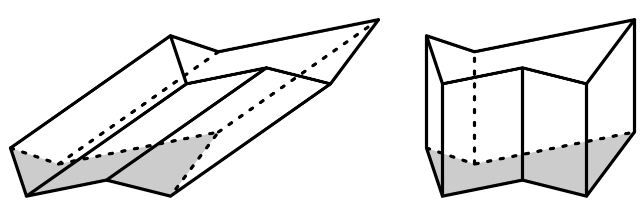


But that's just a cube. What about other 3D shapes? Well, any 3D shape—for example, a horse—can be approximated by many tiny cubes. If we multiply the dimensions of the horse by 1/2 then the volume of each tiny cube gets multiplied by 1/8, hence the volume of the horse gets multiplied by 1/8.

• What is a prism? What is the volume of a prism?

Consider two congruent 2D shapes in space separated by a translation. A **prism** is the 3D shape we get by connecting corresponding points of the 2D shapes. We say that it is a **right prism** if the translation is perpendicular fo the planes of the shapes.

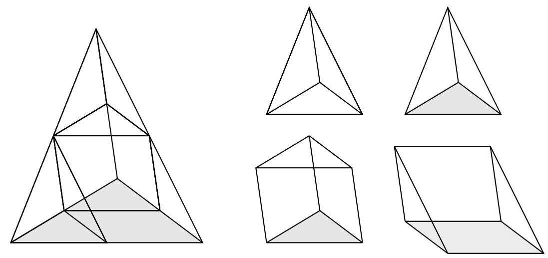
Let B be the area of the 2D shape (called the **base**) and let h be the perpendicular distance between the planes of the two shapes (called the **height**). The volume of the prism is Bh. If we consider two prisms with the same base shape and the same height then **Cavalieri's Principle** says that they should have the same volume. [Think of a stack of coins. It has the same volume even if we build it crooked.] For example, these two prisms have the same volume.



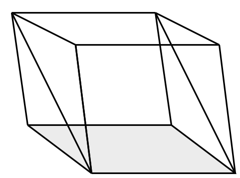
• What is the volume of a tetrahedron?

Consider a tetrahedron where B is the area of the base triangle and h is the perpendicular height. I claim that its volume is $V = \frac{1}{3}Bh$. We will use a dissection described in Euclid's *Elements*, Book XII, Proposition 3.

By connecting the midpoints of the edges we can dissect the tetrahedron into two smaller tetrahedra and two triangular prisms as follows.



Note that the each of the two smaller tetrahedra has volume $\frac{1}{8}V$ because they are similar to the larger tetrahedron but with half the dimensions. One of the triangular prisms is easy to deal with because its base triangle has area $\frac{1}{4}B$ (the base of the large tetrahedron can be divided into four equal triangles) and its perpendicular height is $\frac{1}{2}h$. Thus it has volume $\frac{1}{4}B\frac{1}{2}h = \frac{1}{8}Bh$. The other prism is more difficult because we don't know its height or its base area. **However**, if we double it then we obtain a nicer prism with parallelogram base.



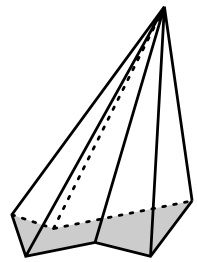
This nicer prism has base with area $\frac{1}{2}B$ (why?) and perpendicular height $\frac{1}{2}h$, thus it has volume $\frac{1}{2}B\frac{1}{2}h = \frac{1}{4}Bh$. Since the nicer prism is double the bad prism, the bad prism must have volume $\frac{1}{2}(\frac{1}{4}Bh) = \frac{1}{8}Bh$. Now we have computed the volumes of the four pieces. Putting them together gives

$$\begin{aligned} V &= \frac{1}{8}V + \frac{1}{8}V + \frac{1}{8}Bh + \frac{1}{8}Bh \\ V &= \frac{1}{4}V + \frac{1}{4}Bh \\ 4V &= V + Bh \\ 3V &= Bh \\ V &= \frac{1}{3}Bh. \end{aligned}$$

Remark: This is the easiest proof I know. Max Dehn proved in 1902 that is it **impossible** to cut a tetrahedron into finitely many pieces and reassemble them to form a cube. Thus we expect that no "finite" proof of the formula $\frac{1}{3}Bh$ is possible. So which part of our proof was not "finite"? [Hint: How did we prove that each smaller tetrahedron has volume $\frac{1}{8}V$?]

• What is the volume of a general cone?

To form a general cone we start with a 2D shape in space and we connect it to some point outside the shape. Just as for the tetrahedron, I claim that the volume is $\frac{1}{3}Bh$ where B is the area of the base shape and h is the perpendicular height.



Let V be the volume of the cone. Suppose you can dissect the base shape into triangles (for example, the base shape shown above can be dissected into four triangles) and let these triangles have areas $B_1, B_2, \ldots, B_n$. Note that we have $B = B_1 + B_2 + \cdots + B_n$. The tetrahedra over these triangles fill up the cone, so we have

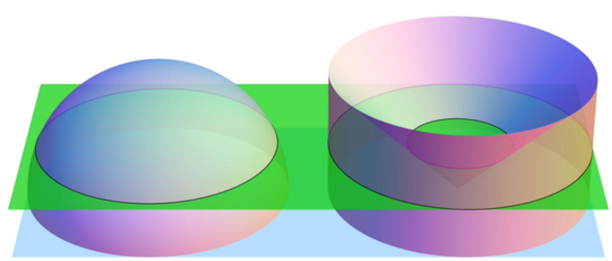
$$\begin{aligned} V &= \frac{1}{3}B_1 h + \frac{1}{3}B_2 h + \cdots + \frac{1}{3}B_n h \\ &= \frac{1}{3}(B_1 + B_2 + \cdots + B_n)h \\ &= \frac{1}{3}Bh. \end{aligned}$$

If the base shape can not be dissected into triangles **exactly** (maybe it's a curvy shape like a circle) then at least we can **approximate** it with many small triangles. In the limit we still get $V = \frac{1}{3}Bh$.

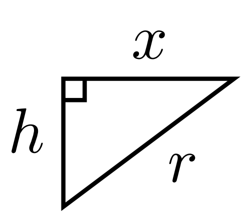
• What is the **volume** of a sphere?

We will show that a sphere of radius r has volume $\frac{4}{3}\pi r^3$. Actually, we will show that a **hemi**sphere of radius r has volume $\frac{2}{3}\pi r^3$.

To do this we will show that the hemisphere has the same volume as a cylinder of height r and radius r minus a cone. Here is a nice picture from the blog of someone named Zachary Abel.



The cross section at height h of the cylinder minus cone is a green annulus with outer circle of radius r and inner circle of radius h. Thus it has area $\pi r^2 - \pi h^2 = \pi(r^2 - h^2)$. The cross section of the hemisphere at height h is a green circle of radius x, hence it has area πx^2. But what is x? Well, the center of the sphere, the center of the green circle, and any point on the boundary of the green circle form a right triangle:

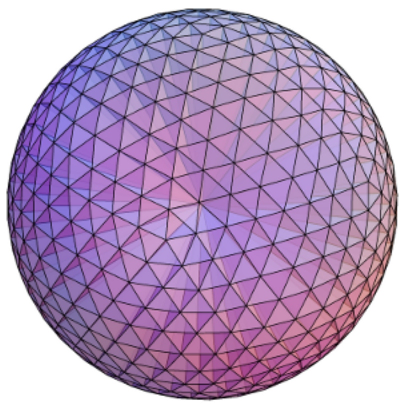


The Pythagoream Theorem (remember that?) says that $h^2 + x^2 = r^2$. Therefore the area of the green circle is $\pi x^2 = \pi(r^2 - h^2)$. Since the hemisphere and the cylinder minus cone have the same cross-sectional areas, Cavalieri's Principle says that they must have the same volume. What is that volume?

Well, I don't yet know the volume of the hemisphere, but I do know that the volume of the cylinder is $(\pi r^2)r = \pi r^3$ (it is a prism after all) and the volume of the cone is $\frac{1}{3}(\pi r^2)r = \frac{1}{3}\pi r^3$. Therefore the volume of the cylinder minus cone (and hence also the volume of the hemisphere) is $\pi r^3 - \frac{1}{3}\pi r^3 = \frac{2}{3}\pi r^3$. We conclude that the volume of the full sphere is $2(\frac{2}{3}\pi r^3) = \frac{4}{3}\pi r^3$.

• Finally, what is the **surface area** of a sphere?

Consider a sphere with radius r. Let S be its surface area. To compute S we will approximate the surface by many small triangles and form a tetrahedron over each of these with vertex at the center of the sphere. Here is another nice picture from Zachary Abel.



Suppose the areas of the triangles are $B_1, B_2, \ldots, B_n$ (where n is very large). Then we have $S \approx B_1 + B_2 + \cdots + B_n$. Let V be the volume of the sphere. Note that each tetrahedron has height approximately equal to r. Since the tetrahedra approximately fill up the sphere we have

$$\begin{aligned} V &\approx \frac{1}{3}B_1 r + \frac{1}{3}B_2 r + \cdots + \frac{1}{3}B_n r \\ &= \frac{1}{3}(B_1 + B_2 + \cdots + B_n)r \\ &\approx \frac{1}{3}Sr. \end{aligned}$$

In the limit this is an equality. On the other hand we alreay know that $V = \frac{4}{3}\pi r^3$. Thus we conclude that

$$\begin{aligned} \frac{1}{3}Sr &= \frac{4}{3}\pi r^3 \\ S &= 4\pi r^2. \end{aligned}$$

Discussion:

• Are you surprised that the surface area of a sphere is harder to compute than its volume? Well, it is. Both the volume and the surface area of a sphere were computed by Archimedes of Syracuse. One can rephrase the result $V = \frac{4}{3}\pi r^3$ by saying that the volume of the sphere is $\frac{2}{3}$ the volume of the smallest cylinder that contains it. Also, the surface area of the sphere is equal to the surface area of the smallest cylinder that contains it (minus the top and bottom circles). Apparently, Archimedes was so pleased with these results he requested that the images of a sphere and a cylinder be placed on his tomb. Cicero reports an expedition to Syracuse in 75 BC (137 years after Archimedes' death) during which he found the tomb abandoned and covered by scrub, but the sphere and cylinder were there and the verses were still partly legible.

• Here is the shortest Calculus proof I know for the surface area of a sphere of radius r. We can think of the volume as an infinite sum of disks. The disk at distance x from the center has area $\pi(r^2 - x^2)$. Thus the volume is

$$\int_{-r}^{r} \pi(r^2 - x^2)\, dx = \pi \left[r^2 x - \frac{1}{3}x^3 \right]_{-r}^{r} = \pi \left[(r^3 - \frac{1}{3}r^3) - (-r^3 + \frac{1}{3}r^3) \right] = \frac{4}{3}\pi r^3.$$

Note that the volume $V(r) = \frac{4}{3}\pi r^3$ is a function of the radius. If we increase the radius by an infinitesimal amount dr, by how much does the volume increase? We can think of the new increase in volume dV as a very thin shell of thickness dr. Because the shell is very thin, it seems plausible to say it has volume $S(r)dr$, where $S(r)$ is the surface area of the sphere. We conclude that $dV = S(r)dr$, and hence

$$S(r) = \frac{dV}{dr} = \frac{4}{3}\pi 3r^2 = 4\pi r^2.$$

This proof is short but not very satisfying. I wouldn't necessarily even believe it if I didn't already know the correct answer. The proof we gave above was longer, but I belived every step because I could see a picture.

So what **is** the benefit of Calculus to geometry? Good question. What is the 4D "hyper-volume" of a 4D "hyperball"? In this case we **can't** draw a picture, but maybe we can still use Calculus. [Remark: We can. The hypervolume is $\frac{1}{2}\pi^2 r^4$.]

• Finally, consider a triangle with angles α, β, γ on the surface of a sphere of radius r. Last time we discussed Thomas Harriot's (1603) proof that the area of this triangle is

$$\frac{S}{4\pi}(\alpha + \beta + \gamma - \pi),$$

where S is the total surface area of the sphere. Now we can finish the calculuation. Substituting $S = 4\pi r^2$ shows that the area of the triangle is

$$r^2(\alpha + \beta + \gamma - \pi).$$

If the sphere is very big ($r \to \infty$) then locally it looks like a flat plane. (Sometimes we say that a flat plane is a sphere of radius ∞.) **Q:** How could we possibly get a triangle with **finite** area on such a sphere? **A:** For the area $r^2(\alpha + \beta + \gamma - \pi)$ to stay finite, the quantity $\alpha + \beta + \gamma - \pi$ must approach zero. That is, the sum of the angles must approach $180°$. This suggest that Euclidean geometry is a limiting case (as $r \to \infty$) of spherical geometry.

By the way, the "surface" of a 4D "hyperball" is a 3D space called a "hypersphere". Maybe our universe is a hypersphere. How could we tell? [Answer: If you left the earth travelling in a straight line, you would eventually end up back where you started. If the distance of the round trip is $2\pi r$, this means that we live in a hypersphere of radius r. It also means that our universe is finite with volume $2\pi^2 r^3$.]