2.7.4. Consider a rectangle with length ℓ, width w and area $A = \ell w$:

Given that $d\ell/dt = +8$ cm/s and $dw/dt = +3$ cm/s we want to find dA/dt. To do this we use the product rule:

$$\frac{dA}{dt} = \frac{d}{dt}(\ell w) = \ell \cdot \frac{dw}{dt} + w \cdot \frac{d\ell}{dt}.$$

When $\ell = 20$ and $w = 10$ we have

$$\frac{dA}{dt} = (20)(3) + (10)(8) = 140 \text{ cm}^2/\text{s}.$$

2.7.6. Consider a sphere with radius r and volume $V = \frac{4}{3}\pi r^3$ (picture omitted). Given that $dr/dt = +4$ mm/s we want to find dV/dt. To do this we use the chain rule:

$$\frac{dV}{dt} = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = \frac{4}{3}\pi \frac{d}{dt}(r^3) = \frac{4}{3}\pi\left(3r^2\frac{dr}{dt}\right).$$

(Note that the constant $4\pi/3$ just comes outside the integral.) When the diameter is 80 we have radius $r = 40$ and hence

$$\frac{dV}{dt} = \frac{4}{3}\pi\left(3r^2\frac{dr}{dt}\right) = \frac{4}{3}\pi \cdot 3(40)^2(4) \approx 80424 \text{ mm}^3/\text{s}.$$

2.8.12. Use linear approximation to find $\sqrt[3]{1001}$. Consider the function $f(x) = \sqrt[3]{x} = x^{1/3}$ with derivative $f'(x) = (1/3)x^{-2/3}$. When $x \approx 1000$ we have

$$\begin{aligned} f(x) &\approx f(1000) + f'(1000)(x - 1000) \\ x^{1/3} &\approx (1000)^{1/3} + \frac{1}{3}(1000)^{-2/3}(x - 1000) \\ x^{1/3} &\approx 10 + \frac{1}{300}(x - 1000). \end{aligned}$$

Substituting $x = 1001$ gives

$$1001^{1/3} \approx 10 + \frac{1}{300}(1001 - 1000) = 10 + \frac{1}{300} = 10.0033333333\cdots.$$

(The correct value is $1001^{1/3} = 10.003332222839094952\cdots$.)

2.8.14. Use linear approximation to find $1/4.002$. Consider the function $f(x) = 1/x = x^{-1}$ with derivative $f'(x) = (-1)x^{-2} = -1/x^2$. When $x \approx 4$ we have

$$f(x) \approx f(4) + f'(4)(x - 4)$$

$$\frac{1}{x} \approx \frac{1}{4} - \frac{1}{4^2}(x-4).$$

Substituting $x = 4.002$ gives

$$\frac{1}{4.002} \approx \frac{1}{4} - \frac{1}{4^2}(4.002-4) = \frac{1}{4} - \frac{1}{16}(0.002) = 0.250125.$$

(The correct value is $1/4.002 = 0.24987506246876561719\cdots$.)

2.8.22. The radius of a disk is given as 24 cm with error 0.2 cm.

(a) Use differentials to estimate the error in the calculated area of the disk. The area is $A = \pi r^2$ with differential dA given by

$$\frac{dA}{dr} = \pi(2r) \quad \rightsquigarrow \quad dA = 2\pi r dr.$$

Substituting $r = 24$ and $dr = 0.2$ gives

$$A = \pi(24)^2 \approx 1809.56,$$
$$dA = 2\pi(24)(0.2) \approx 30.16.$$

Hence the area of the disk is 1809.56 plus or minus 30.16 cm^2.

(b) The percentage error is $30.16/1809.56 = 0.017\%$.

2.8.24. Use differentials to estimate the amount of paint needed to apply a coat of paint 0.05 cm thick to a hemispherical dome with diameter 50 m.

This is a tricky one. I think it's easier to solve without Calculus.

Solution Without Calculus. Paint is measured by volume. The unpainted hemisphere of radius 25 meters has volume $(2/3)\pi r^3 = (2/3)\pi(25)^3 = 32724.923$ meters^3. The painted hemisphere has radius 25.0005 meters (because the paint is 0.5 centimeters, or 0.0005 meters, thick), so the painted hemisphere has volume $(2/3)\pi(25.0005)^3 = 32726.887$ meters^3. The difference of these volumes is the volume of the paint:

$$\text{volume of paint} = 32724.887 - 32726.923 = 1.967 \text{ meters}^3.$$

That is a lot of paint! Approximately 520 gallons.

Solution With Calculus. The radius is $r = 25$. The added paint increases the radius by a tiny amount $dr = 0.0005$. The volume of the hemisphere is $V = (2/3)\pi r^3$. The paint increases this volume by a tiny amount dV, where

$$\frac{dV}{dr} = \frac{d}{dr}\left(\frac{2}{3}\pi r^3\right) = \frac{2}{3}\pi(3r^2) \quad \rightsquigarrow \quad dV = 2\pi r^2 dr.$$

Substituting $r = 25$ and $dr = 0.0005$ gives the approximate volume of paint:

$$dV \approx 2\pi(25)^2(0.005) = 1.963 \text{ meters}^3.$$

That's very close to the exact answer computed above.

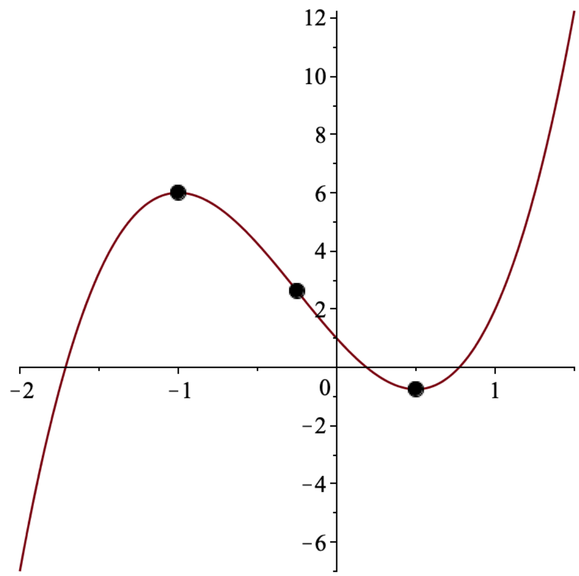
3.3.2. We will compute the first and second derivatives of $f(x) = 4x^3 + 3x^2 - 6x + 1$ and use this information to sketch the graph. The first derivative is

$$f'(x) = 4(3x^2) + 3(2x) - 6(1) = 12x^2 + 6x - 6 = 6(2x^2 + x - 1) = 6(x+1)(2x-1).$$

The second derivative is

$$f''(x) = 12(2x) + 6(1) = 6(4x+1).$$

The first derivative is zero when $x = -1$ or $x = 1/2$. It is positive (f is increasing) when $x < -1$ or $x > 1/2$ and negative (f is decreasing) when $-1 < x < 1/2$. The second derivative is zero when $x = -1/4$. It is positive (f is concave up) when $x > -1/4$ and negative (f is concave down) when $x < -1/4$. There is an inflection point at $(1/2, -3/4)$, a local maximum at $(-1, 6)$ and a local minimum at $(-1/4, 21/8)$. Here is a picture (not to scale):



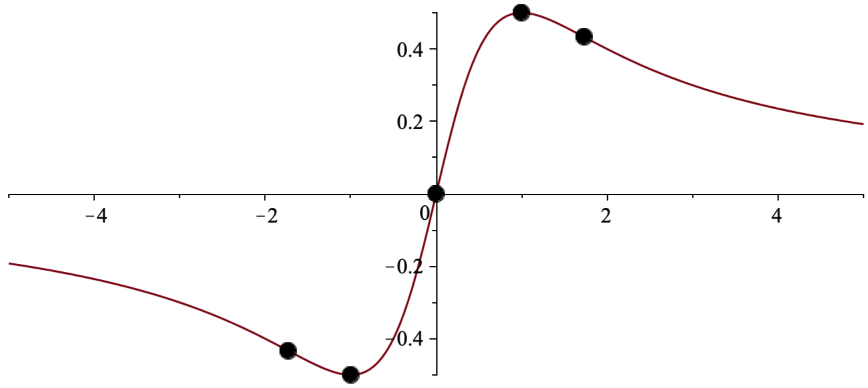
3.3.4. We will compute the first and second derivatives of $f(x) = x/(x^2+1)$ and use this information to sketch the graph. The first derivative is

$$f'(x) = \frac{(x^2+1)(1) - (x)(2x+0)}{(x^2+1)^2} = \frac{-x^2+1}{(x^2+1)^2} = \frac{(1-x)(1+x)}{(x^2+1)^2}.$$

The second derivative is

$$f''(x) = \frac{(x^2+1)^2(-2x) - (-x^2+1)[2(x^2+1)(2x+0)]}{(x^2+1)^4} = \cdots = \frac{2x(x^2-3)}{(x^2+1)^3}.$$

The first derivative is zero when $x = 1$ or $x = -1$. It is negative (f is increasing) when $x < -1$ or $x > 1$ and negative (f is decreasing) when $-1 < x < 1$. The second derivative is zero when $x = 0$ or $x = \pm 3$. It is positive (f is concave up) when $0 < x < -\sqrt{3}$ or $x > \sqrt{3}$ and negative (f is concave down) when $x < -\sqrt{3}$ or $0 < x < \sqrt{3}$. There are inflection points when $x = -\sqrt{3}, 0, \sqrt{3}$, a local minimum when $x = -1$ and a local maximum when $x = 1$. Here is a picture (not to scale):



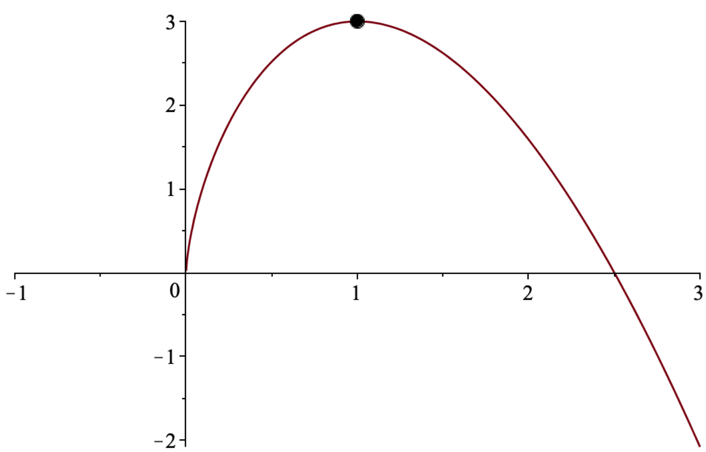
3.3.28. We will compute the first and second derivatives of $G(x) = 5x^{2/3} - 2x^{5/3}$ and use this information to sketch the graph. (Note that this function is only defined when $x \geq 0$.) The first derivative is

$$G'(x) = 5 \cdot \frac{2}{3}x^{-1/3} - 2 \cdot \frac{5}{3}x^{2/3} = \cdots = -\frac{10}{3} \cdot \frac{x-1}{x^{1/3}}.$$

The second derivative is

$$G''(x) = 5 \cdot \frac{2}{3} \cdot \frac{-1}{3}x^{-4/3} - 2 \cdot \frac{5}{3} \cdot \frac{2}{3}x^{-1/3} = \cdots = -\frac{10}{9} \cdot \frac{2x+1}{x^{4/3}}.$$

The first derivative is zero when $x = 1$. It is positive (G is increasing) when $0 < x < 1$ and negative (G is decreasing) when $x > 1$. Since $x \geq 0$, the second derivative is always negative, hence G is always concave down. There is a local maximum when $x = 1$. I guess you could also say there is a local minimum when $x = 0$, but that point is a bit strange. The tangent becomes vertical as $x \to 0^+$ because $\lim_{x\to 0^+} G'(x) = +\infty$. Here is a picture (not to scale):



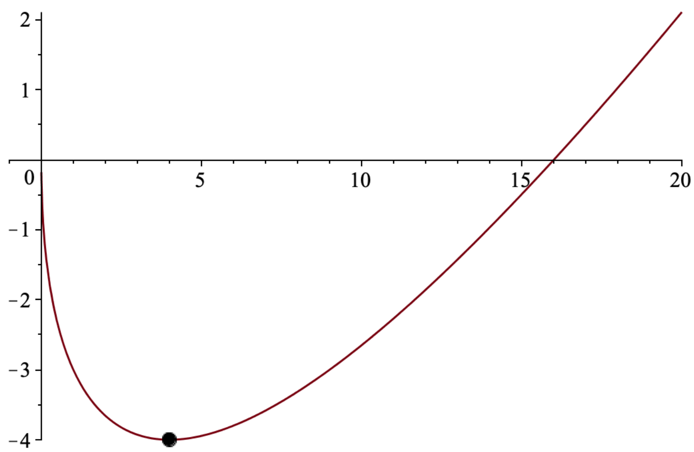
3.3.30. We will compute the first and second derivatives of $G(x) = x - 4\sqrt{x}$ and use this information to sketch the graph. (Note that this function is only defined when $x \geq 0$.) The first derivative is

$$G'(x) = 1 - 4 \cdot \frac{1}{2\sqrt{x}} = 1 - 2x^{-1/2}.$$

The second derivative is

$$G''(x) = 0 - 2 \cdot (-1/2)x^{-3/2} = \frac{1}{x^{3/2}}.$$

The first derivative is zero when $x = 4$. It is positive (G is increasing) when $x > 4$ and negative (G is decreasing) when $0 < x < 4$. Since $x \geq 0$, the second derivative is always positive, hence G is always concave up. There is a local minimum when $x = 4$. I guess you could also say there is a local maximum when $x = 0$, but that point is a bit strange. The tangent becomes vertical as $x \to 0^+$ because $\lim_{x\to 0^+} G'(x) = -\infty$. Here is a picture (not to scale):



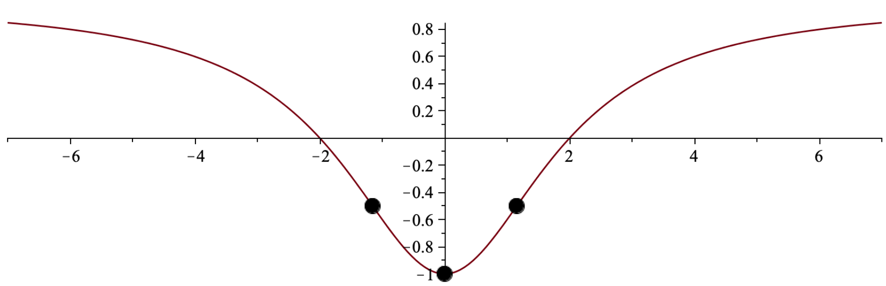
3.3.34. We will compute the first and second derivatives of $f(x) = (x^2 - 4)/(x^2 + 4)$ and use this information to sketch the graph. The first derivative is

$$f'(x) = \frac{(x^2+4)(2x+0) - (x^2-4)(2x+0)}{(x^2+4)^2} = \cdots = \frac{16x}{(x^2+4)^2}.$$

The second derivative is

$$f''(x) = \frac{(x^2+4)^2(16) - (16x)[2(x^2+4)(2x+0)]}{[(x^2+4)^2]^2} = \cdots = \frac{16(-3x^2+4)}{(x^2+4)^3}.$$

The first derivative is zero when $x = 0$. It is negative (f is decreasing) when $x < 0$ and positive (f is increasing) when $x > 0$. The second derivative is zero when $x = \pm\sqrt{4/3}$. It is positive (f is concave up) when $-\sqrt{4/3} < x < \sqrt{4/3}$ and negative (f is concave down) when $x < -\sqrt{4/3}$ or $x > \sqrt{4/3}$. There are inflection points when $x = \pm\sqrt{4/3}$ and a local minimum when $x = 0$. There is a horizontal asymptote at $y = 1$ because $\lim_{x\to\pm\infty} f(x) = 1$. Here is a picture (not to scale):



3.5.2. Find two numbers whose difference is 100 and whose product is a minimum. Call the numbers x and y. We are given that $x - y = 100$ and we want to minimize the product $P(x, y) = xy$. Since we don't know how to deal with multivariable functions (that is the topic of Calculus 3) we will use the constraint to write $y = x - 100$ and hence P is a function of x alone:

$$P(x) = xy = x(x - 100) = x^2 - 100x.$$

To minimize P we set the first derivative equal to zero:

$$\begin{aligned} P'(x) &= 0 \\ 2x - 100 &= 0 \\ x &= 50. \end{aligned}$$

We conclude that P is minimized when $x = 50$, and hence $y = x - 100 = 50 - 100 = -50$.

3.5.4. The sum of two numbers is 16. What is the smallest possible value of the sum of their squares? Call the numbers x and y. We are given $x + y = 16$ and we want to minimize the function $S(x, y) = x^2 + y^2$. Since we don't know how to deal with multivariable functions we will use the constraint to write $y = 16 - x$ and hence S is a function of x alone:

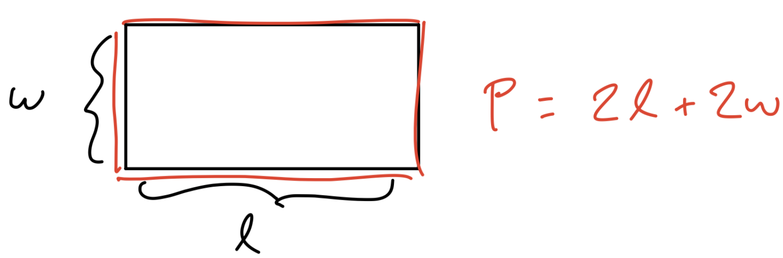
$$S(x) = x^2 + y^2 = x^2 + (16 - x)^2.$$

To minize S we set the first derivative equal to zero:

$$\begin{aligned} S'(x) &= 0 \\ 2x + 2(16 - x)(0 - 1) &= 0 \\ 2x - 2(16 - x) &= 0 \\ 2x - 32 + 2x &= 0 \\ 4x &= 32 \\ x &= 8. \end{aligned}$$

We conclude that S is maximized when $x = 8$, and hence $y = 16 - 8 = 8$.

3.5.8. Find the dimensions of a rectangle with area 1000 m^2 whose perimeter is as small as possible. If ℓ and w are the dimensions of the rectangle then the perimeter is $P = 2\ell + 2w$:



We want to minimize $P(\ell, w) = 2\ell + 2w$ subject to the constraint $\ell w = 1000$. First we use this constraint to eliminate w from P:

$$P(\ell) = 2\ell + 2w = 2\ell + 2(1000/\ell) = 2\ell + 2000/\ell.$$

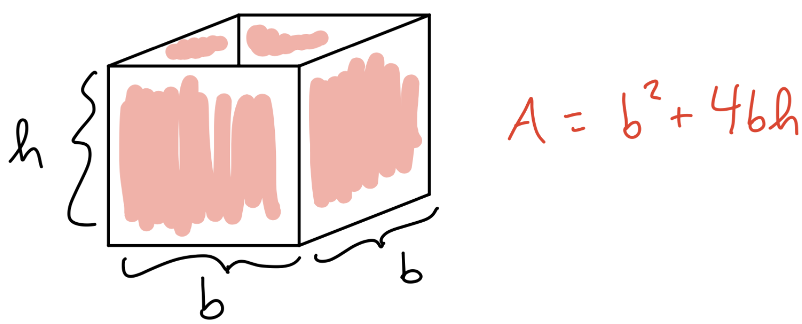
Then to minimize P we set the first derivative equal to zero:

$$\begin{aligned} P'(\ell) &= 0 \\ 2 + 2000(-1/\ell^2) &= 0 \end{aligned}$$

$$
\begin{aligned}
-2000/\ell^2 &= -2 \\
1/\ell^2 &= 2/2000 \\
\ell^2 &= 1000 \\
\ell &= \sqrt{1000}.
\end{aligned}
$$

We conclude that P is maximized when $\ell = \sqrt{1000}$, and hence $w = 1000/\sqrt{1000} = \sqrt{1000}$. In other words, for a given area the perimeter is maximized when the rectangle is a square.

3.5.12. (Oops, this problem was not assigned. So you can call it a practice problem.) A box with a square base and open top must have a volume of 32000 cm^3. Find the dimensions of the box that minimize the amount of material used (say, cardboard). Let b be the base and let h be the height of the box. The amount of cardboard is the surface area $A = b^2 + 4bh$:



In order to minimize A we first eliminate h using the volume constraint:

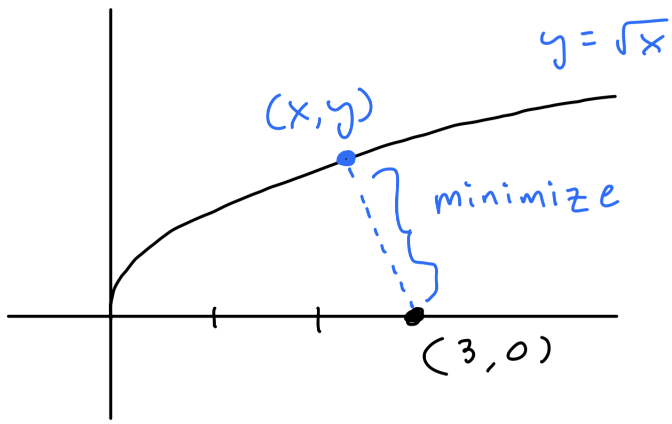
$$
\begin{aligned}
\text{volume} &= 32000 \\
b^2 h &= 32000 \\
h &= 32000/b^2.
\end{aligned}
$$

Hence we have $A = b^2 + 4bh = b^2 + 4b(32000/b^2) = b^2 + 128000/b$. Then to minimize A we set the first derivative equal to zero:

$$
\begin{aligned}
A'(b) &= 0 \\
2b + 128000(-1/b^2) &= 0 \\
2b^3 - 128000 &= 0 \\
2b^3 &= 128000 \\
b^3 &= 64000 \\
b &= 40.
\end{aligned}
$$

We conclude that the amount of material is minimized when $b = 40$ and $h = 32000/40^2 = 20$.

3.5.16. Find the point (x, y) on the curve $y = \sqrt{x}$ that is closest to the point $(3, 0)$:



The distance between **any two points** (x, y) and (a, b) is $\sqrt{(x-a)^2+(y-b)^2}$. In particular, the distance between (x, y) and $(3, 0)$ is $D = \sqrt{(x-3)^2+(y-0)^2} = \sqrt{x^2-6x+9+y^2}$. In order to minimize the distance we first use the constraint $y = \sqrt{x}$ to eliminate y from D:

$$D = \sqrt{x^2-6x+9+y^2} = \sqrt{x^2-6x+9+x} = \sqrt{x^2-5x+9}.$$

Now we set the first derivative equal to zero:

$$\begin{aligned} D'(x) &= 0 \\ \frac{1}{2\sqrt{x^2-5x+9}}(2x-5+0) &= 0 \\ 2x-5 &= 0 \\ x &= 5/2. \end{aligned}$$

(Here we used the fact that $a/b = 0$ implies $a = 0$ for any fraction.) We conclude that the distance D is minimized when $x = 5/2$ and $y = \sqrt{x} = \sqrt{5/2}$.

3.6.8. Use Newton's method with initial guess $x_1 = -1$ to find a root of the equation $x^7 + 4 = 0$, correct to four decimal places.
We want to solve the equation $f(x) = 0$ with $f(x) = x^7 + 4$. Newton's method gives the recursive equation

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^7+4}{7x_n^6}.$$

Starting from $x_1 = -1$ we have

n	x_n
1	−1.000
2	−1.4286
3	−1.2917
4	−1.2302
5	−1.2193
6	−1.2190
7	−1.2190

3.6.14. Use Newton's method to find the positive root of $3\sin x = x$, correct to six decimal places. We want to solve the equation $f(x) = 0$ where $f(x) = 3\sin x - x$. Newton's method gives the recursive equation

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{3\sin(x_n) - x_n}{3\cos(x_n) - 1}.$$

In this case the initial guess $x_1 = 1$ leads to the wrong solution, so we need to be more careful. After looking at the graph on Desmos I see that the solution is close to 2, so I'll guess $x_1 = 2$. Then we have

n	x_n
1	2.000000
2	2.323732
3	2.279595
4	2.278863
5	2.278863

A.1. The Babylonian Algorithm for Square Roots.

(a) In order to compute $\sqrt{a}$ we will solve the equation $f(x) = 0$ where $f(x) = x^2 - a$. Newton's method gives the recursive equation

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\ &= x_n - \frac{x_n^2 - a}{2x_n} \\ &= \frac{2x_n^2 - (x_n^2 - a)}{2x_n} \\ &= \frac{x_n^2 + a}{2x_n} \\ &= \frac{1}{2}\left(\frac{x_n^2 + a}{x_n}\right) \\ &= \frac{1}{2}\left(x_n + \frac{a}{x_n}\right). \end{aligned}$$

This is called the Babylonian algorithm.

(b) In order to compute $\sqrt{3}$ we let $a = 3$. Then starting with the guess $x_1 = 1$ we obtain

n	x_n
1	1.000000
2	2.000000
3	1.750000
4	1.732143
5	1.732051
6	1.732051