Math 161 **Summer 2023**
Exam 3 Solutions **Drew Armstrong**

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1. The area A of a square is increasing at a constant rate of 1 cm^2/sec. At what rate is the side length increasing when $A = 4\ \text{cm}^2$?

Let ℓ be the side length of the square, so that $A = \ell^2$. Using the chain rule we have

$$\frac{dA}{dt} = 2\ell \cdot \frac{d\ell}{dt} \quad \rightsquigarrow \quad \frac{d\ell}{dt} = \frac{1}{2\ell} \cdot \frac{dA}{dt} = \frac{1}{2\ell} \cdot (1) = \frac{1}{2\ell}.$$

When $A = 4$ then we have $\ell = \sqrt{4} = 2$ and hence

$$\frac{d\ell}{dt} = \frac{1}{2(2)} = \frac{1}{4}\ \text{cm/sec}.$$

2. Use linear approximation to estimate the value of $\sqrt{9.06}$.

For any differentiable function $f(x)$ and number a we have

$$x \approx a \quad \rightsquigarrow \quad f(x) \approx f(a) + f'(a)(x-a).$$

Taking $f(x) = \sqrt{x}$ and $a = 9$ gives

$$x \approx 9 \quad \rightsquigarrow \quad \sqrt{x} \approx \sqrt{9} + \frac{1}{2\sqrt{9}}(x-9) = 3 + \frac{1}{6}(x-9).$$

Since $x = 9.06$ is close to 9, this implies that

$$\sqrt{9.06} \approx 3 + \frac{1}{6}(9.06-9) = 3 + \frac{1}{6}(0.06) = 3.01.$$

3. The radius of a disk is measured to be 5 cm with a maximum error of 0.1 cm. What is the maximum error in the computed area of the disk?

Let r be the radius of the disk, so the area of the disk is $A = \pi r^2$. We are given that $r = 5$ and $dr = 0.1$, which implies that

$$dA = 2\pi r\, dr = 2\pi(5)(0.1) = \pi.$$

Remark: Since $A = \pi(5)^2 = 25\pi$ this gives a percentage error of $\pi/(25\pi) = 1/25 = 4\%$. You were not asked to compute this.

4. Find the point on the line $2x + y = 1$ that is closest to the point $(0,0)$. Hint: The distance between $(0,0)$ and a general point (x,y) is $\sqrt{x^2+y^2}$.

We want to minimize the distance function $D(x,y) = \sqrt{x^2+y^2}$ subject to the constraint $2x + y = 1$. First we use the constraint equation $y = 1 - 2x$ to eliminate y from D:

$$D(x) = \sqrt{x^2 + (1-2x)^2}.$$

Then we set the derivative equal to zero:

$$\begin{aligned} D'(x) &= 0 \\ \frac{1}{2\sqrt{x^2+(1-2x)^2}}[2x + 2(1-2x)(-2)] &= 0 \\ 2x + 2(1-2x)(-2) &= 0 \end{aligned}$$

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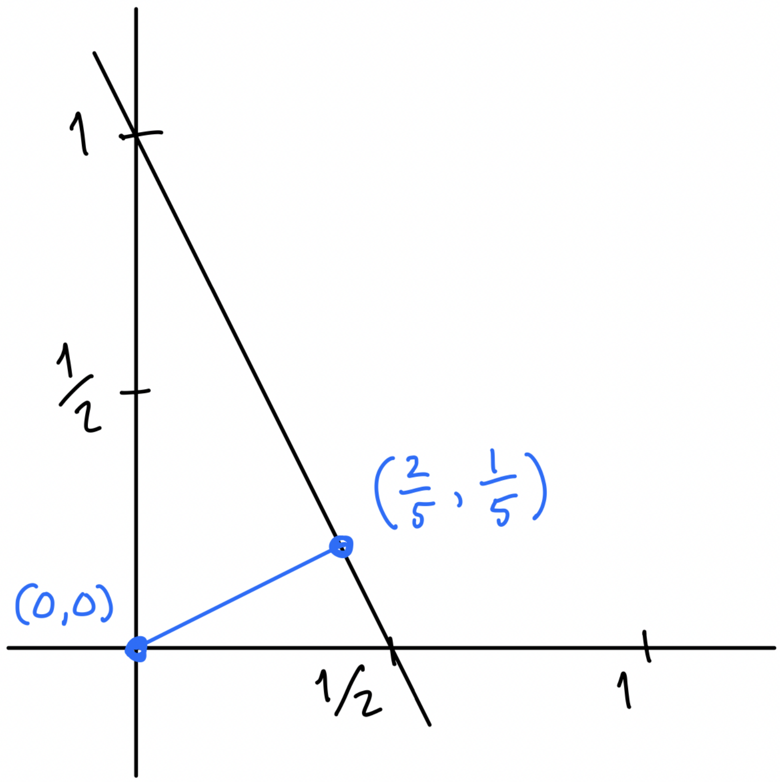
\begin{aligned}
2x - 4 + 8x &= 0 \\
10x - 4 &= 0 \\
x &= 4/10 \\
x &= 2/5, \\
y &= 1 - 2(2/5) = 1/5.
\end{aligned}
$$

Thus $(x, y) = (2/5, 1/5)$ is the desired point. Picture:



5. What is the recursive equation that Newton's method will use to compute $\sqrt{5}$?

To solve the equation $f(x) = 0$, Newton's method uses the recursive equation

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

In our case we wish to solve $f(x) = 0$ where $f(x) = x^2 - 5$, so we will use

$$x_{n+1} = x_n - \frac{x_n^2 - 5}{2x_n}.$$

Remark: This formula can be rearranged to give the "Babylonian algorithm formula":

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{5}{x_n}\right).$$