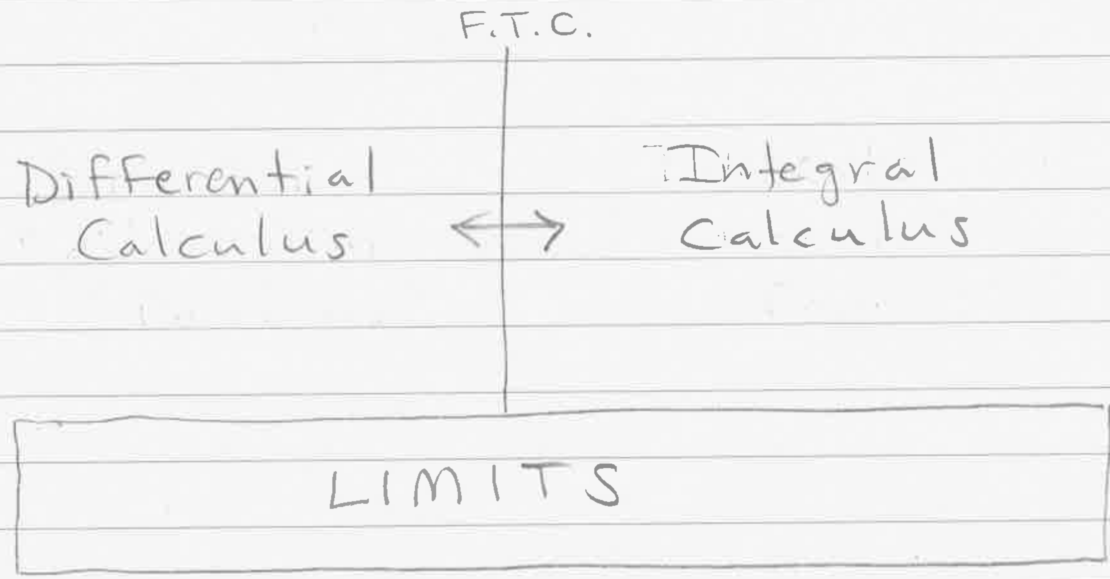
7/20/15

We are now halfway through the course.
(Today is the 15th class out of 27.)

Recall the birds-eye view of Calculus:



So far we have covered

- Chap 1 : Limits
- Chap 2 & 3 : Differential Calculus

Now it is time to dive into

- Chap 4 : Integral Calculus.

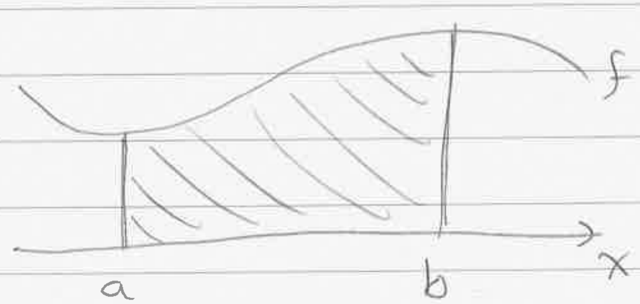
This will bring back the ideas from the first two days of class.

Integral Calculus is all about computing areas & volumes. At first this seems to have nothing to do with computing derivatives. But in the 1660s and 1670s, Isaac Newton and Gottfried Leibniz discovered that derivatives & integrals are somehow "opposites". This discovery (called the Fundamental Theorem of Calculus) is the heart of the subject. We'll discuss the F.T.C. this week.

Let $f(x)$ be a function of a real variable x. Recall that the expression

$$\int_a^b f(x)\,dx$$

refers to the area between the graph of $f(x)$ and the x-axis, from $x=a$ to $x=b$.

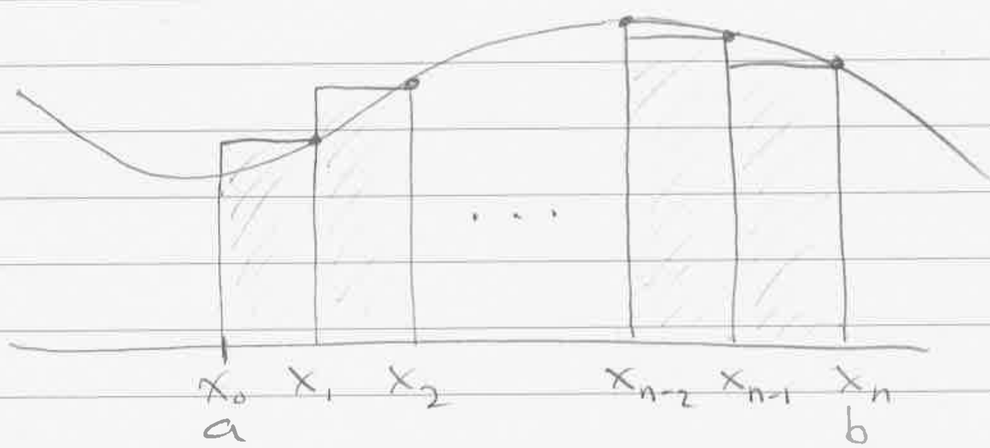


shaded area

$$= \int_a^b f(x)\,dx.$$

OK, but this notation is fairly worthless unless we have some way to <u>compute</u> the area.

The first idea (going back to Archimedes and earlier) is to <u>approximate</u> the region with many small shapes whose areas we know. The standard way to do this is to divide the interval $[a,b]$ into n equal subintervals.



where $x_0 = a$ and $x_n = b$.

Problem: Find a formula for x_i.

The width of $[a,b]$ is $b-a$, so the width of each rectangle should be .

$$\Delta x = \frac{b-a}{n}$$

Then we find

$$x_1 = x_0 + \Delta x$$
$$x_2 = x_1 + \Delta x = x_0 + 2\Delta x$$
$$x_3 = x_2 + \Delta x = x_0 + 3\Delta x$$
$$\vdots$$

$$\boxed{x_i = x_0 + i \cdot \Delta x}$$

Is this formula correct? Check:

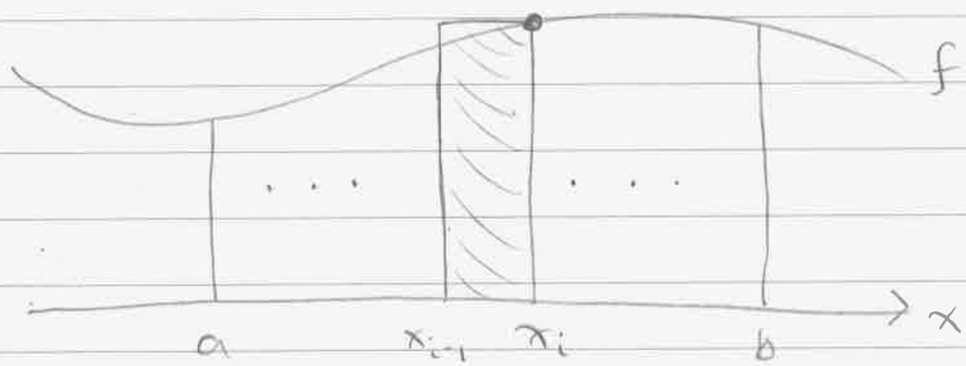
$$x_0 = x_0 + 0 \cdot \Delta x \quad \checkmark$$

$$x_n = x_0 + n \cdot \Delta x$$
$$= a + n\left(\frac{b-a}{n}\right)$$
$$= \cancel{a} + (b - \cancel{a}) = b \quad \checkmark.$$

Yes this is correct.

Problem: what is the height of the ith rectangle?

The i^{th} rectangle is based on the interval $[x_{i-1}, x_i]$



We could compute the height based on <u>any</u> x-value in this interval. For now let's just choose the right hand endpoint $x = x_i$. So the height of the ith rectangle is $f(x_i)$.

Hence the area of the ith rectangle is

$$(\text{height}) \times (\text{base}) = f(x_i) \cdot \Delta x$$

The total area of the n rectangles is

$$f(x_1) \cdot \Delta x + f(x_2) \cdot \Delta x + \cdots + f(x_n) \cdot \Delta x.$$

There is a convenient shorthand notation for this called "sigma notation"

$$f(x_1)\cdot\Delta x + f(x_2)\cdot\Delta x + \cdots + f(x_n)\cdot\Delta x$$

$$= \sum_{i=1}^{n} f(x_i)\cdot\Delta x$$

= "The sum of $f(x_i)\cdot\Delta x$ as i goes from 1 to n."

Let's give a name to this. We'll call it

$$R_n := \sum_{i=1}^{n} f(x_i)\cdot\Delta x$$

The idea is that R_n is approximately equal to the area under the curve from $x=a$ to $x=b$, and that the approximation gets better as $n \longrightarrow \infty$.

In fact, this will be our definition of the integral.

Definition: Consider a function $f(x)$ and an interval $[a,b]$. Let $\Delta x = (b-a)/n$ and consider the sum

$$R_n = \sum_{i=1}^{n} f(x_i)\cdot\Delta x.$$

Then we define the area under the graph from $x=a$ to $x=b$ as

$$\int_a^b f(x)\,dx := \lim_{n\to\infty} R_n$$

///

We may or may not be able to compute this value, but at least we've defined it.

Remarks:

- The quantity $\int_a^b f(x)\,dx$ is called the integral of $f(x)$ from $x=a$ to $x=b$.

  Just like the derivative of $f(x)$, it is defined as a limit. (However, the definition of the integral looks much more complicated.)

- The notation $\int_a^b f(x)\,dx$ was invented by Leibniz and it is supposed to remind us of the definition:

$$\underbrace{\int_a^b f(x)\,dx}_{\text{sum the areas of infinitely many infinitesimal rectangles.}} = \lim_{n\to\infty} \underbrace{\left[\sum_{i=1}^{n} f(x_i)\cdot \Delta x \right]}_{\text{sum the areas of finitely many skinny rectangles}}$$

The symbols $\int$ and $\sum$ are both versions of the letter "S", for sum.

- Isaac Newton called $\int f(x)\,dx$ the <u>Fluent</u> of $f(x)$ and he used the notation

$$\overset{|}{f}(x) \quad \text{or} \quad \boxed{f(x)}$$

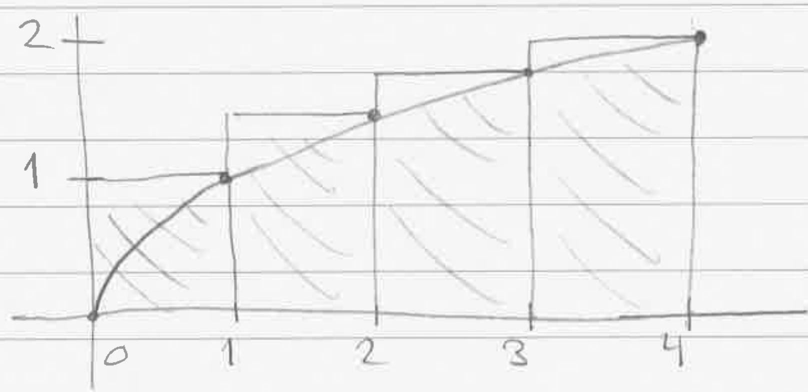
Compare to his notation $\dot{f}(x) = f'(x)$ for <u>fluxions</u>. Newton's notation for integrals is completely obsolete.

Practice: Chap 4.1 Exercise 4.

Estimate the area under the graph of $f(x) = +\sqrt{x}$ from $x=0$ to $x=4$ using 4 rectangles

(a) first using right endpoints.
(b) then using left endpoints

(a)



We have $a=0$, $b=4$, $n=4$, so

$$\Delta x = (b-a)/n = (4-0)/4 = 1.$$

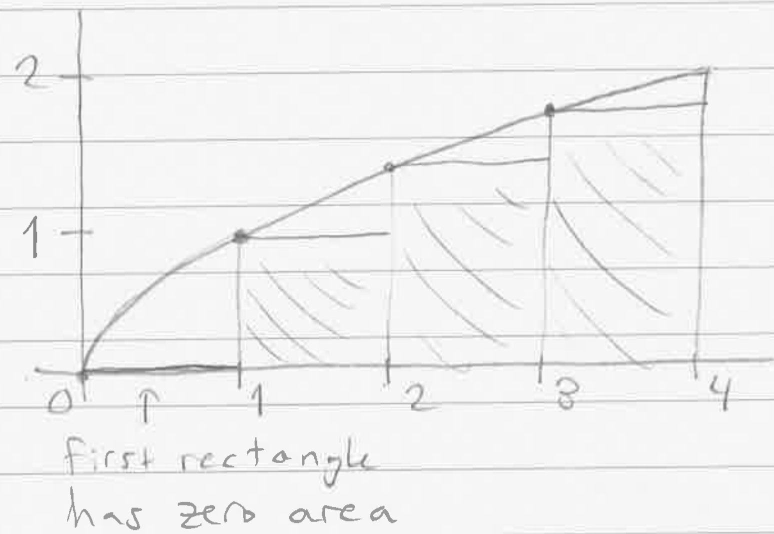
The total area of the 4 rectangles is

$$R_4 = f(1)\Delta x + f(2)\Delta x + f(3)\Delta x + f(4)\Delta x$$

$$= \sqrt{1}\cdot 1 + \sqrt{2}\cdot 1 + \sqrt{3}\cdot 1 + \sqrt{4}\cdot 1$$

$$= 6.146.$$

(b)



Here $\Delta x = 1$, just as before. The total area of the 4 rectangles is

$$L_4 = f(0)\cdot\Delta x + f(1)\cdot\Delta x + f(2)\cdot\Delta x + f(3)\cdot\Delta x$$

L is for "left hand endpoints"

$$= \sqrt{0}\cdot 1 + \sqrt{1}\cdot 1 + \sqrt{2}\cdot 1 + \sqrt{3}\cdot 1$$

$$= 4.146$$

Note that R_4 was an overestimate and L_4 was an underestimate. The actual area is in between:

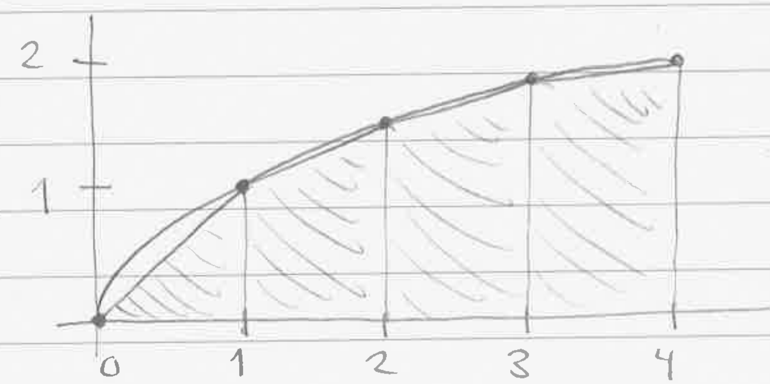
$$4.146 < \int_0^4 \sqrt{x}\,dx < 6.146$$

The average of these two estimates is

$$\frac{L_4 + R_4}{2} = \frac{4.146 + 6.146}{2}$$

$$= 5.146.$$

This represents the area of four "trapezoids":



As you see, it's still a bit of an underestimate. To get the true value we must compute the limit

$$\int_0^4 \sqrt{x}\, dx = \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{x_i} \cdot \Delta x$$

where $\Delta x = (4-0)/n = 4/n$ and $x_i = 0 + i \cdot \Delta x = 4i/n$.

We have

$$R_n = \sum_{i=1}^{n} \sqrt{x_i} \cdot \Delta x$$

$$= \sum_{i=1}^{n} \sqrt{\frac{4i}{n}} \cdot \frac{4}{n}$$

$$= \sum_{i=1}^{n} \frac{2\sqrt{i}}{\sqrt{n}} \cdot \frac{4}{n}$$

$$= \frac{8}{n^{3/2}} \sum_{i=1}^{n} \sqrt{i}$$

and hence

$$\int_0^4 \sqrt{x}\, dx = \lim_{n \to \infty} R_n$$

$$= 8 \cdot \left[\lim_{n \to \infty} \frac{1}{n^{3/2}} \sum_{i=1}^{n} \sqrt{i} \right]$$

We do <u>not</u> have the technology to evaluate this limit yet, but we will soon.

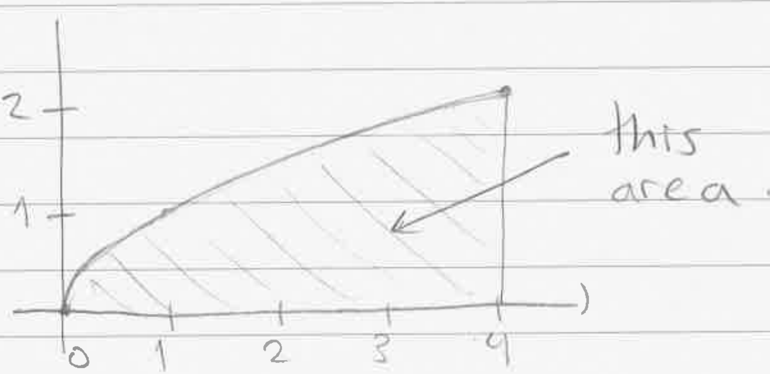
7/21/15

Quiz 3	Total	10
	Ave	6.2
	Med	6.0
	St Dev	2.4

HW 4 due on Friday.

Yesterday we considered the integral

$$\int_0^4 \sqrt{x}\, dx =$$



By using approximations we found that the area is slightly more than 5.146

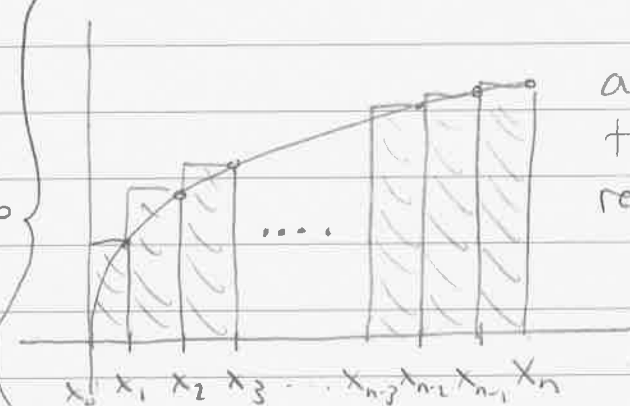
The exact value is defined as a limit:

$$\int_0^4 \sqrt{x}\, dx = \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{x_i} \cdot \Delta x$$

where $\Delta x = (4-0)/n = 4/n$ and
$x_i = 0 + i \cdot \Delta x = 4i/n$.

Here's the Picture:

$$\int_0^4 \sqrt{x}\, dx = \lim_{n \to \infty}$$



area of the n rectangles

Here's the Algebra:

$$\int_0^4 \sqrt{x}\, dx = \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{\frac{4i}{n}} \cdot \frac{4}{n}$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} \frac{2\sqrt{i}}{\sqrt{n}} \cdot \frac{4}{n}$$

$$= 8 \cdot \lim_{n \to \infty} \sum_{i=1}^{n} \frac{i^{1/2}}{n^{3/2}}.$$

OK, fine. But this limit is actually <u>really</u> hard to compute.

The miracle of Calculus is that there is a sneaky trick for calculating integrals that avoids the problem of evaluating horrible limits.

The sneaky trick is called the

Fundamental Theorem of Calculus.

It was discovered by Newton (in the 1660s) and Leibniz (in the 1670s) and it is really the heart of the subject.

Before explaining the F.T.C. I'll just give you a glimpse of how it works. Think of it like a magic trick.

To compute $\int_a^b f(x)\,dx$:

First find a function $F(x)$ such that

$$F'(x) = f(x).$$

We call $F(x)$ an anti<u>deriv</u>ative of $f(x)$.

Then the integral is given by

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a).}$$

That's it.

So let's apply this to compute $\int_0^4 \sqrt{x}\,dx$.

First we want to find $F(x)$ such that

$$F'(x) = \sqrt{x}.$$

Any guesses?

Let's guess that there is a solution of the form $F(x) = c \cdot x^p$ for constants c and p. Then

$$x^{1/2} = F'(x) = c \cdot p\, x^{p-1}.$$

This implies $p - 1 = 1/2$, hence $p = 3/2$.
And then $cp = 1$ implies $c = 2/3$.

$\int$

So we guess that $F(x) = \frac{2}{3} x^{3/2}$.

Does it work? Check:

$$F'(x) = \frac{\cancel{2}}{\cancel{3}} \cdot \frac{\cancel{3}}{\cancel{2}} x^{3/2 - 1} = x^{1/2} \quad \checkmark$$

Great, we found an antiderivative for $\sqrt{x}$. Now the F.T.C. tells us that

$$\int_0^4 \sqrt{x}\, dx = F(4) - F(0)$$
$$= \frac{2}{3}(4)^{3/2} - \cancel{\frac{2}{3}(0)^{3/2}}$$
$$= \frac{2}{3} \cdot 8 - 0$$
$$= 16/3$$
$$= 5.333\ldots$$

Wow that was a lot easier than evaluating the limit of a crazy sum. It is kind of like magic.

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Example:

Compute $\int_a^b x^2\,dx$ using the F.T.C.

First we need to find $F(x)$ such that

$$F'(x) = x^2.$$

Any guesses? Use the same trick as before: Guess that $F(x) = c \cdot x^p$ for some c & p. Then compute

$$1 \cdot x^2 = F'(x) = c \cdot p\, x^{p-1}.$$

$$p - 1 = 2 \implies p = 3$$
$$c \cdot p = 1 \implies c = 1/3.$$

We guess that $F(x) = \frac{1}{3} x^3$. Check:

$$F'(x) = \frac{1}{\cancel{3}} \cdot \cancel{3} x^2 = x^2. \qquad \checkmark$$

Summary:

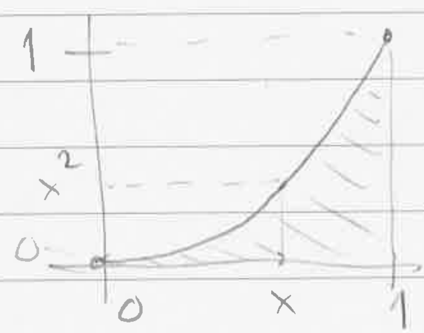
$\frac{1}{3} x^3$ is an antiderivative of x^2.

Now use the F.T.C. to get

$$\int_a^b x^2\,dx = F(b) - F(a)$$
$$= \frac{1}{3}b^3 - \frac{1}{3}a^3$$
$$= \frac{1}{3}\left(b^3 - a^3\right).$$

Special Case: Let $a=0$ and $b=1$ to get

$$\int_0^1 x^2\,dx = \frac{1}{3}\left(1^3 - 0^3\right) = \frac{1}{3}.$$



area $= 1/3$.

Recall that we proved this on the 2nd day of class, but then we used a <u>much</u> <u>harder</u> method.

///

While we're at it, let's pause to record a general rule.

We've seen that:

- $\frac{2}{3}x^{3/2}$ is an antiderivative of $x^{1/2}$
- $\frac{1}{3}x^{3}$ is an antiderivative of x^{2}

In general we have the following rule.

☆ Antiderivative of a Power.

Let p be any constant except -1. Then

$$\boxed{\frac{1}{p+1}x^{p+1} \text{ is an antiderivative of } x^{p}}$$

Proof: $\left(\frac{1}{p+1}\cdot x^{p+1}\right)' = \frac{1}{\cancel{p+1}}\cancel{(p+1)}x^{(p+1)-1}$

$$= 1 \cdot x^{p}.$$

[Puzzle: So what is an antiderivative of x^{-1}? Just wait. We'll see later.]

Now let me try to explain <u>why</u> the F.T.C. is true. To do this we need to think about the integral

$$\int_a^b f(x)\,dx$$

as a function of a & b. Certainly if we change the values of a & b then the area under f between $x=a$ and $x=b$ changes.

[Remark: Notice that $\int_a^b f(x)\,dx$ is NOT a function of x. The x in this expression is just a dummy variable. We could replace it by any letter we want.

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(s)\,ds.$$]

First let's write down some basic properties:

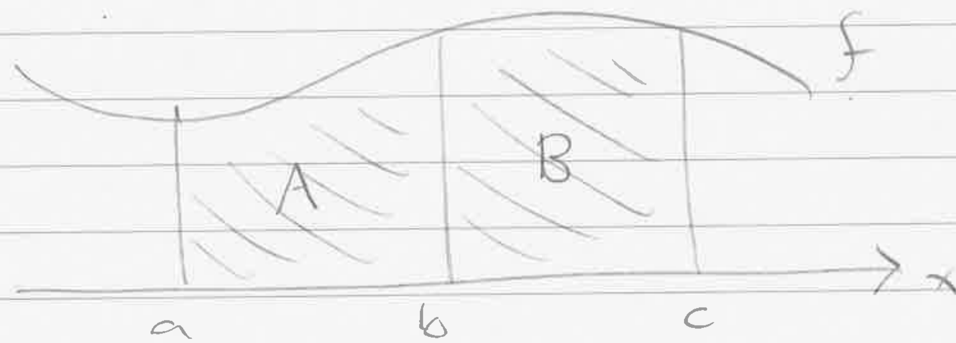
- $\int_a^a f(x)\,dx = 0$ for any a.

EASY.

• Given $a < b < c$ we have

$$\boxed{\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx} \quad (*)$$

Proof: Look at the following picture.



$$\text{area } A = \int_a^b f(x)\,dx$$

$$\text{area } B = \int_b^c f(x)\,dx$$

$$\text{area } A + \text{area } B = \int_a^c f(x)\,dx.$$

//

Now let's try something weird/abstract. Let's pretend that formula $(*)$ is true for <u>all</u> values of a, b, c. (i.e. not only when $a \le b \le c$).

For example, let a & b be any numbers and let $c = a$. Then formula (*) says

$$\int_a^a f(x)\,dx = \int_a^b f(x)\,dx + \int_b^a f(x)\,dx.$$

$$0 = \int_a^b f(x)\,dx + \int_b^a f(x)\,dx$$

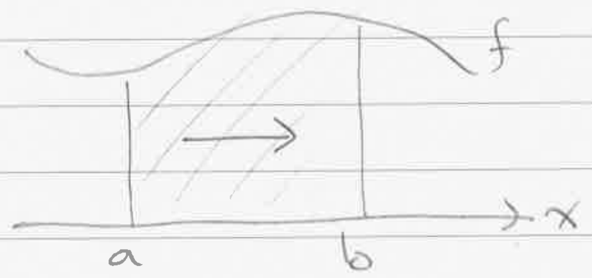
$$\boxed{\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx}$$

Wait a minute. Does this make any sense?

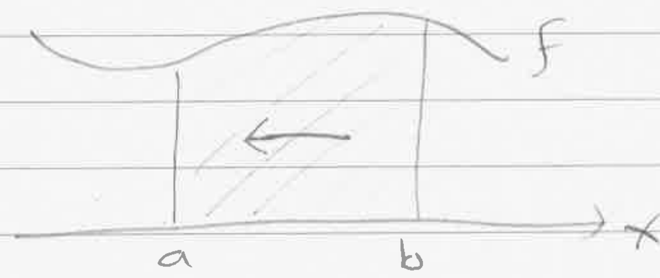
Problem: Is there such a thing as "negative area"?

Maybe not, but for the purposes of computing integrals it is incredibly useful to pretend that there is.

By convention we will call the area under $f(x)$ negative when we read it from right to left:



POSITIVE AREA

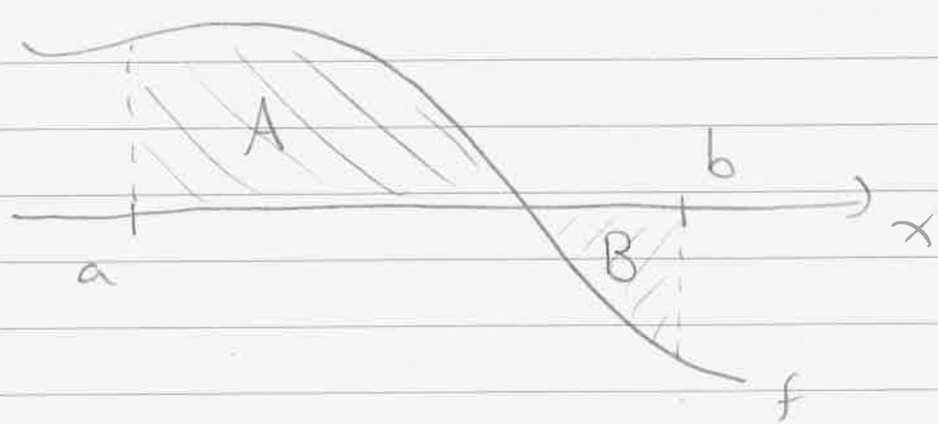


NEGATIVE AREA.

Yes it's weird but we need to do this to make the calculations easier. Now formula (*) is true for all a, b, c, and we like that.

In fact, there is another way for negative area to occur.

Problem: Consider the graph of the following function $f(x)$.



Which of the following is correct?

1. $\int_a^b f(x)\,dx$ = area A + area B.

2. $\int_a^b f(x)\,dx$ = area A − area B.

You might prefer 1 but in fact we must choose 2 to make the calculations work out right.

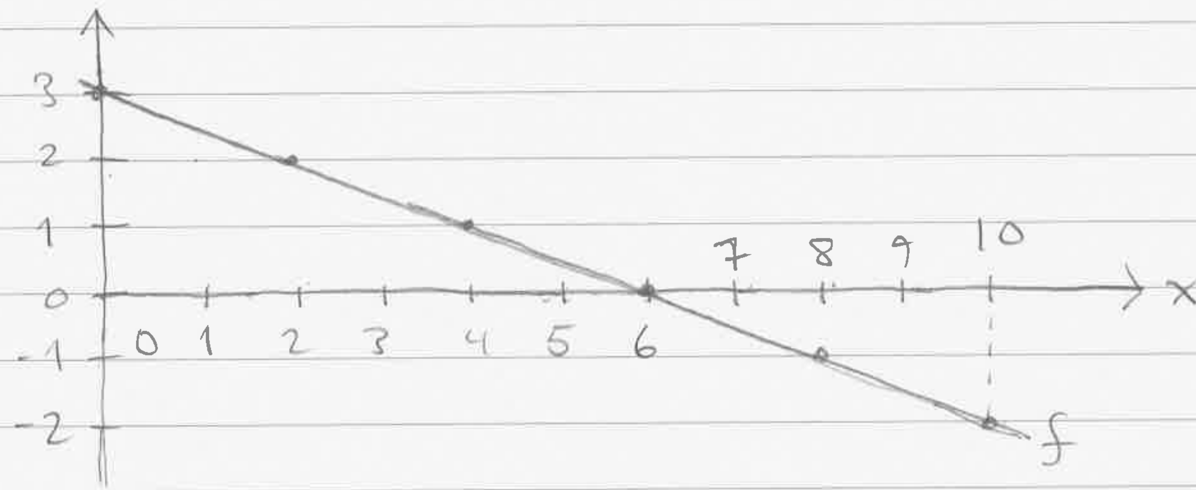
☆ Convention: Area below the x-axis counts as negative area.

So the integral doesn't really represent area, it represents a certain kind of "signed area". After we do the calculations we should look at the picture to make sure we interpret the negative signs correctly.

Now let's practice.

Problem: Consider the function

$$f(x) = 3 - \frac{1}{2}x.$$



Compute the following integrals

(a) $\displaystyle\int_0^{10} f(x)\,dx$ (b) $\displaystyle\int_0^{6} f(x)\,dx$

(c) $\displaystyle\int_6^{10} f(x)\,dx$ (d) $\displaystyle\int_2^{10} f(x)\,dx$

(e) $\displaystyle\int_{10}^{6} f(x)\,dx$

Here we don't really need the F.T.C. because the regions are made out of triangles whose area we know.

(a) $\displaystyle\int_0^{10} f(x)\,dx = \frac{6 \cdot 3}{2} - \frac{4 \cdot 2}{2}$

$= 9 - 4 = 5.$

(b) $\displaystyle\int_0^{6} f(x)\,dx = \frac{6 \cdot 3}{2} = 9.$

(c) $\displaystyle\int_6^{10} f(x)\,dx = -\frac{4 \cdot 2}{2} = -4$

(d) $\displaystyle\int_2^{10} f(x)\,dx = \frac{4 \cdot 2}{2} - \frac{4 \cdot 2}{2} = 0$

[Here the positive and negative area canceled to give zero.]

(e) $\displaystyle\int_{10}^{6} f(x)\,dx = -\int_6^{10} f(x)\,dx$

$= -(-4) = +4.$

[Wrap your head around that one!]

Tomorrow we will see that all this nonsense about negative area is worth it because we need it to make the Fundamental Theorem work properly.

Stay tuned . . .

7/22/15

HW4 due Friday.

Yesterday we discussed the Fundamental Theorem of Calculus:

"Let $f(x)$ and $F(x)$ be functions such that $F'(x) = f(x)$. Then for all a & b we have

$$\int_a^b f(x)\,dx = F(b) - F(a).\text{"}$$

And we saw that it can do some miraculous calculations.

Then we discussed the general properties of

$$\int_a^b f(x)\,dx$$

thought of as a function of a & b. By interpreting "negative area" in a suitable way, we decided that the formula

$\downarrow$

(*) $$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$

is true for any numbers a, b & c. Now we will use this formula to see why the F.T.C. is true.

BOLD IDEA:

Since $\int_a^b f(x)\,dx$ is a function of a and b, maybe we could compute its derivative?

Remark: Why would we want to? So far we have a very poor understanding of this function and we don't even know how to compute it in most cases. Trying to compute its derivative seems like sheer madness! [see Spivak pg. 102].

OK, but there is a fine line between sheer madness and sheer genius, so let's just try it.

To get a function of one variable we should fix one of a or b. Let's fix a and think of b as a variable. Then

$$A(b) = \int_a^b f(x)\,dx$$

is a function of b alone. Now lets be brave and try to compute $A'(b) = dA/db$.

By definition we have

$$A'(b) = \lim_{h \to 0} \frac{A(b+h) - A(b)}{h}$$

$$= \lim_{h \to 0} \frac{\int_a^{b+h} f(x)\,dx - \int_a^b f(x)\,dx}{h}$$

Now what? We can "simplify" the numerator using our general formula (*).

First use $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ to get

$$\int_a^{b+h} f(x)\,dx - \int_a^b f(x)\,dx = \int_a^{b+h} f(x)\,dx + \int_b^a f(x)\,dx$$

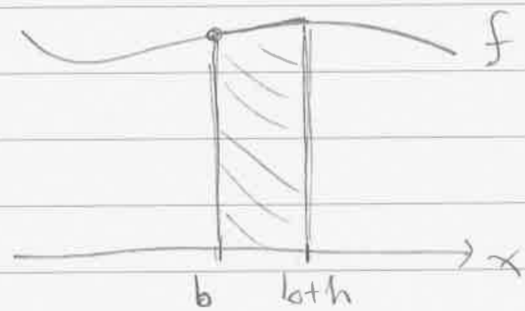
Then use (*) to get

$$\int_a^{b+h} f(x)\,dx + \int_b^a f(x) = \int_b^{b+h} f(x)\,dx .$$

Thus we have

$$A'(b) = \lim_{h \to 0} \frac{\int_b^{b+h} f(x)\,dx}{h}.$$

Now what? Let's draw a picture of this region:



As $h \to 0$ this skinny region looks more and more like a <u>rectangle</u>.

The base of the "rectangle" is h and the height of the "rectangle" is $\approx f(b)$, so

$$\int_b^{b+h} f(x)\,dx \approx h \cdot f(b) \text{ when } h \approx 0.$$

Since $A'(b)$ is the limit as $h \to 0$ we can replace $\int_b^{b+h} f(x)\,dx$ by $h \cdot f(b)$ to get

$$A'(b) = \lim_{h \to 0} \frac{\cancel{h} \cdot f(b)}{\cancel{h}} = f(b).$$

Yay! Wait, what did we do?

We just computed the derivative of the incredibly complicated function

$$A(b) = \int_a^b f(x)\,dx$$

and found that the answer is quite simple:

$$\frac{d}{db} A(b) = \frac{d}{db} \int_a^b f(x)\,dx = f(b)$$

In other words, the function $A(b)$ is an antiderivative of the function $f(b)$!

["A" is for "area" & "antiderivative.
Coincidence?]

Maybe you still don't appreciate why this is good so let me put it another way.

We want to compute the area

$$\int_a^b f(x)\,dx$$

but it seems really hard. Now we know that the area function

$$A(b) = \int_a^b f(x)\,dx$$

is just an antiderivative of the function f. So if we know the antiderivatives of f this will allow us to compute the area.

Summary:

$$\text{Computing an Integral} \xleftrightarrow{\text{F.T.C.}} \text{Computing an Antiderivative.}$$

///

Now we really care about antiderivatives so we should think more about them.

Back to Chapter 3.7 !

Problem: We have seen that $x^3/3$ is an antiderivative of x^2. Why was I careful not to call it the antiderivative?

Because it's not unique. Check that $x^3/3 + 1$ is another antiderivative of x^2.

$$\left(\frac{1}{3}x^3 + 1\right)' = \frac{1}{\cancel{3}} \cdot \cancel{3}x^2 + 0 = x^2 \checkmark.$$

In fact, any function of the form

$$\frac{1}{3}x^3 + c$$

where c is a constant is an antiderivative of x^2.

OK, fine. Does x^2 have any other antiderivatives? NO.

☆ Theorem (pg. 189 of Stewart).

Let $f(x)$ be a function. If $F(x)$ is any particular antiderivative of $f(x)$, then the general antiderivative of $f(x)$ is

$$F(x) + C,$$

where C is an arbitrary constant.

For example, let $f(x) = x^p$ where p is any constant except -1. Then the most general antiderivative of $f(x)$ is

$$\frac{1}{p+1} \cdot x^{p+1} + C.$$

Back to the F.T.C., we know that

$$A(b) = \int_a^b f(x)\,dx$$

is an antiderivative of the function $f(b)$. Which one is it?

Suppose $F(b)$ is another antiderivative of $f(b)$, so the <u>most</u> <u>general</u> antiderivative is

$$F(b) + c$$

where c is a constant. Since $A(b)$ is another antiderivative we must have

$$\boxed{A(b) = F(b) + c}. \quad (**)$$

How can we compute c? Since

$$A(b) = \int_a^b f(x)\,dx$$

we have the important fact that

$$A(a) = \int_a^a f(x)\,dx = 0.$$

Plugging $b = a$ into $(**)$ gives

$$F(a) + c = A(a) = 0$$

$$\Longrightarrow c = -F(a).$$

and we conclude that

$$A(b) = F(b) - F(a).$$

In other words,

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$$

Yes, this is the magic formula we used yesterday. The subtlety and abstraction of the ideas is compensated for by the fact that this formula is incredibly useful.

If you followed those ideas, great! If not, don't worry too much. It's more important that you know how to <u>use</u> the ideas to compute integrals.

So let's practice that.

Recitation: Evaluate the following integrals from the Chap 4.3 Exercises.

1. $\int_{-2}^{3} (x^2 - 3)\,dx$

7. $\int_{0}^{\pi} (4\sin\theta - 3\cos\theta)\,d\theta$

9. $\int_{1}^{4} \left(\frac{4+6u}{\sqrt{u}}\right) du$

1. Let $f(x) = x^2 - 3$. One particular antiderivative of this is

$$F(x) = \frac{1}{3}x^3 - 3x.$$

Check:

$$F'(x) = \frac{1}{\cancel{3}} \cdot \cancel{3}x^2 - 3 \cdot 1 = x^2 - 3 \quad \checkmark$$

Now we can apply the F.T.C.

$$\int_{-2}^{3} (x^2-3)\,dx = \int_{-2}^{3} f(x)\,dx$$

$$= F(3) - F(-2)$$

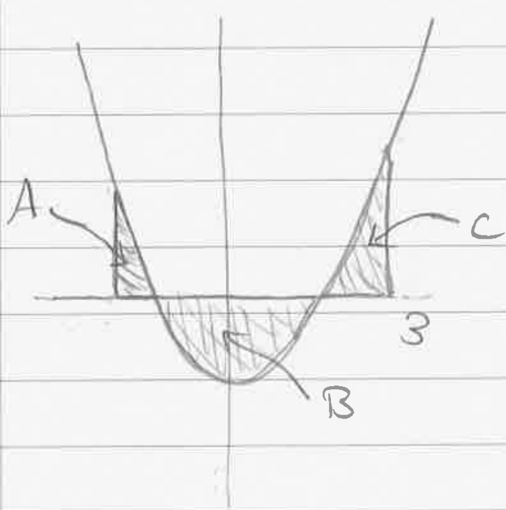
$$= \left[\tfrac{1}{3}(3)^3 - 3(3)\right] - \left[\tfrac{1}{3}(2)^3 - 3(2)\right]$$

$$= [9-9] - \left[\tfrac{8}{3} - 6\right]$$

$$= 0 - \frac{10}{3}$$

$$= -\frac{10}{3} = -3.33\cdots$$

What? The area is negative. Let's see the picture:



$$\int_{-2}^{3} (x^2-3)\,dx$$

$$= \text{area } A + \text{area } C - \text{area } B$$

$$= -10/3.$$

//

7. $\int_0^{\pi} (4\sin\theta - 3\cos\theta)\, d\theta$.

Let $f(\theta) = 4\sin\theta - 3\cos\theta$. One particular antiderivative of this is

$$F(\theta) = -4\cos\theta - 3\sin\theta.$$

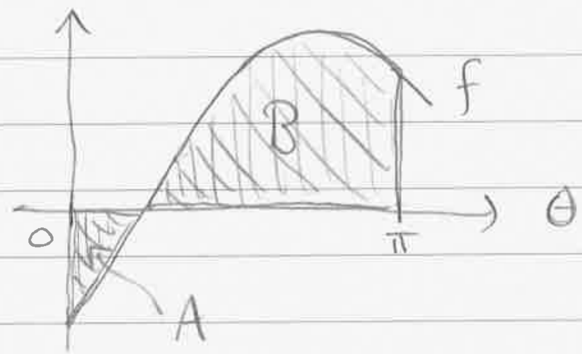
Check:

$$F'(\theta) = -4(-\sin\theta) - 3(\cos\theta)$$
$$= +4\sin\theta - 3\cos\theta \quad \checkmark.$$

Then the F.T.C. says

$$\int_0^{\pi} (4\sin\theta - 3\cos\theta)\, d\theta = F(\pi) - F(0)$$
$$= \left[-4\cos(\pi) - 3\sin(\pi)\right]$$
$$- \left[-4\cos(0) - 3\sin(0)\right]$$
$$= \left[-4(-1) - 3(0)\right] - \left[-4(1) - 3(0)\right]$$
$$= 4 + 4 = 8.$$

Here is the picture:



$$\int_0^{\pi} f(\theta)\,d\theta = \text{area } B - \text{area } A = 8.$$

9. $\displaystyle\int_1^4 \left(\frac{4+6u}{\sqrt{u}}\right) du.$

Let $f(u) = (4+6u)/\sqrt{u}$. Before computing an antiderivative we must put it in a form we recognize.

$$f(u) = \frac{4+6u}{\sqrt{u}} = \frac{4}{\sqrt{u}} + \frac{6u}{\sqrt{u}}$$

$$= \frac{4}{u^{1/2}} + \frac{6u}{u^{1/2}} = 4u^{-1/2} + 6u^{1/2}.$$

Now we compute an antiderivative.

$$F(u) = 4\frac{u^{1/2}}{1/2} + 6\frac{u^{3/2}}{3/2}$$

$$= 4 \cdot \frac{2}{1} u^{1/2} + 6 \cdot \frac{2}{3} u^{3/2}$$

$$= 8u^{1/2} + 4u^{3/2}.$$

Then the F.T.C. says

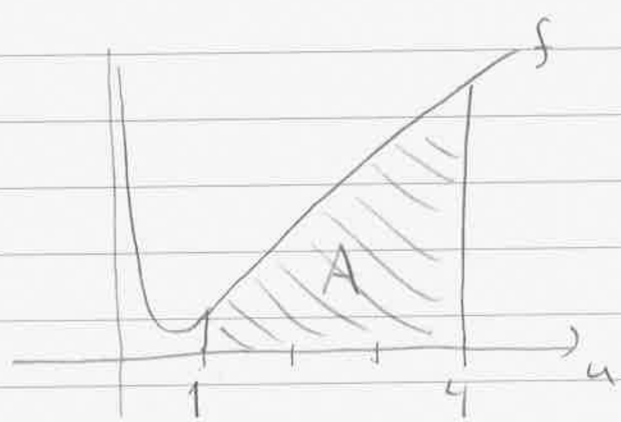
$$\int_1^4 \left(\frac{4+6u}{\sqrt{u}}\right) du = F(4) - F(1)$$

$$= \left[8(4)^{1/2} + 4(4)^{3/2}\right] - \left[8(1)^{1/2} + 4(1)^{3/2}\right]$$

$$= \left[8 \cdot 2 + 4 \cdot 8\right] - \left[8 \cdot 1 + 4 \cdot 1\right]$$

$$= \left[16 + 32\right] - \left[8 + 4\right] = 36.$$

Picture:



$$\int_1^4 \left(\frac{4+6u}{\sqrt{u}}\right) du$$

$$= \text{area } A = 36.$$

///

7/23/15

HW 4 due tomorrow.
Office Hours today 1-2pm
Quiz 4 Monday

Yesterday we thoroughly discussed the Fundamental Theorem of Calculus. This is simultaneously the most important and the most subtle idea in the whole subject.

It will take time to absorb the F.T.C. Hopefully HW4 will help.

Here is the F.T.C. as stated in the text.

☆ The Fundamental Theorem of Calculus:

Let $f(x)$ be a function.

1. If $g(x) = \int_a^x f(t)\,dt$ then $g'(x) = f(x)$
2. $\int_a^b f(x)\,dx = F(b) - F(a)$

where F is any <u>anti</u>derivative of f.

Part (2) looks just as it did in class yesterday, but I stated part (1) differently.

Recall that $\int_a^b f(x)\,dx$ is a function of a & b. If we let <u>a be fixed</u> and compute the derivative with respect to b we get

$$\boxed{\frac{d}{db}\left[\int_a^b f(x)\,dx\right] = f(b).}$$

In my experience people have a hard time looking at this formula, but it's not really so bad.

Examples:

- Let $g(x) = \int_7^x \sqrt{\sin\theta}\cdot d\theta$. Compute $g'(x)$.

Ok, let $f(\theta) = \sqrt{\sin\theta}$. Then we want

$$g'(x) = \frac{d}{dx}\left[\int_7^x f(\theta)\,d\theta\right] = ?$$

Yeah it looks complicated but the formula immediately gives the answer:

$$\frac{d}{dx}\int_7^x f(\theta)\,d\theta = f(x) = \sqrt{\sin x}\,.$$

Nothing to it.

- Here's a "trick question".

Let $g(x) = \int_{x^2}^{9} H(v)\,dv$. Compute $g'(x)$.

This problem is trying to confuse you by mixing up a lot of symbols. Remain calm and think it through.

First get the x on top:

$$g(x) = -\int_9^{x^2} H(v)\,dv\,.$$

Now make the substitution $u = x^2$.

$$g(x) = -\int_9^{u} H(v)\,dv\,.$$

Now use the chain rule.

$$\frac{dg}{dx} = \frac{dg}{du} \cdot \frac{du}{dx}$$

Since $u = x^2$ we have $du/dx = 2x$.
To compute dg/du we can apply Part (1) of the F.T.C.

$$\frac{dg}{du} = \frac{d}{du}\left[-\int_9^u H(v)\,dv\right]$$

$$= -\frac{d}{du}\int_9^u H(v)\,dv$$

$$= -H(u).$$

Finally we have

$$g'(x) = \frac{dg}{dx} = \frac{dg}{du}\frac{du}{dx}$$

$$= -H(u)\cdot(2x)$$

$$= -H(x^2)\cdot 2x.$$

Those kinds of problems aren't good for much except fooling students on exams.

P.S. I won't ask you any question like this on an exam because I think they're silly. But I did want to desensatize you a bit. :)

Recall that I first introduced the derivative by talking about position, velocity & acceleration.

Now we're ready to go back and apply integrals/antiderivatives to physics.

If $s(t)$ is the height of an apple at time t, then $v(t) = s'(t)$ is the velocity and $a(t) = v'(t) = s''(t)$ is the acceleration of the apple at time t.

Galileo & Newton tell us that gravity = acceleration. In particular, if the apple is close to the surface of the earth, and if gravity is the only force acting on the apple, then we can assume

$$\boxed{a(t) = -32 \text{ feet/sec}^2}$$

The problem now is to compute $s(t)$ and predict what the apple will do.

Well, $a(t) = v'(t)$. So $v(t)$ is an antiderivative of $a(t) = -32$. The most general antiderivative is

$$-32t + c$$

where c is an arbitrary constant. Thus we have

$$v(t) = -32t + c.$$

What is c in this case? Plus in $t=0$ to get

$$v(0) = -32 \cdot 0 + c = c.$$

Hence $v(t) = v(0) - 32t$,

where $V(0)$ is the velocity of the apple at time $t=0$ (i.e. the "initial velocity").

Now since $s'(t) = v(t)$, $s(t)$ is an antiderivative of $v(t)$. The most general antiderivative of $v(t) = v(0) - 32t$ is

$$v(0) \cdot t - 32 \cdot \frac{1}{2} t^2 + d \,,$$

where d is another arbitrary constant. Hence

$$s(t) = v(0) \cdot t - 16t^2 + d.$$

To compute d we set $t=0$ to get

$$s(0) = \cancel{v(0) \cdot 0} - \cancel{16(0)^2} + d = d.$$

we conclude that

$$\boxed{s(t) = s(0) + v(0) \cdot t - 16t^2} \,,$$

where $s(0)$ is the initial height and $v(0)$ is the initial velocity of the apple.

Practice: Example from pg. 193 of Stewart.

A ball is thrown upward with speed 48 ft/s from the edge of a cliff 432 ft above the ground.

- Find its height above the ground t seconds later.
- When does it reach its maximum height?
- When does it hit the ground?

Let $s(t)$ = height of ball at time t seconds.
We have $s(0) = 432$ feet.

Let $v(t)$ = velocity of ball at time t seconds.
We have $v(0) = +48$ ft/s.

Let $a(t)$ = acceleration of ball at time t.
By the laws of physics & experiment, we have

$$a(t) = -32 \text{ ft/s}^2.$$

☆ Now I'm going to give you permission to use a special notation. Instead of saying "the most general antiderivative of $a(t)$",

↓

we'll just write "$\int a(t)\,dt$."

Convention: An integral without limits will stand for the general antiderivative. It does not necessarily refer to an area. Please use responsibly.

We have

$$v(t) = \int a(t)\,dt$$

$$= \int -32\,dt = -32t + c$$

Putting $t=0$ gives

$$v(0) = -32 \cdot 0 + c$$
$$48 = 0 + c$$

$$\Longrightarrow \quad v(t) = 48 - 32t.$$

Then we have

$$s(t) = \int v(t)\,dt$$

$$= \int (48 - 32t)\,dt$$

$$= 48t - 32 \cdot \frac{1}{2} t^2 + d.$$

Putting $t=0$ gives

$$s(0) = 48\cdot 0 - 16\cdot 0^2 + d.$$
$$432 = 0 - 0 + d.$$

$$\Longrightarrow \quad s(t) = 432 + 48t - 16t^2. \quad ///$$

This is the height at time t. When is the height maximized?

Answer: $s(t)$ is maximized when $s'(t) = 0$.

Solve $s'(t) = v(t) = 0$.

$$48 - 32t = 0$$
$$48 = 32t$$
$$48/32 = t$$
$$3/2 = t$$

We know this is a maximum because $s''(3/2) = a(3/2) = -32 < 0$.

The ball reaches maximum height when

$$t = 3/2 = 1.5 \text{ seconds}.$$

When does the ball hit the ground?

Answer: When $s(t) = 0$.

Solve $-16t^2 + 48t + 432 = 0$

$$t^2 - 3t - 27 = 0$$

This doesn't factor easily so we use the quadratic formula to get

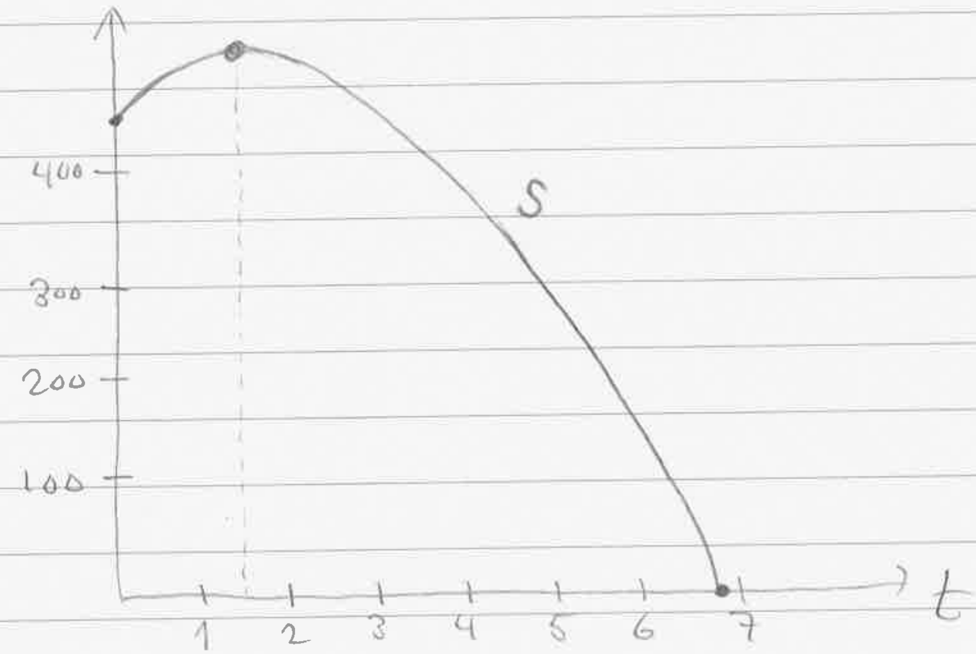
$$t = \frac{-(-3) \pm \sqrt{(-3)^2 - 4 \cdot 1 \cdot (-27)}}{2 \cdot 1}$$

$$= \frac{3 \pm \sqrt{117}}{2}$$

$$= \cancel{-3.91} \text{ or } +6.91 \text{ seconds.}$$

We disregard negative time, so the ball hits the ground at 6.91 seconds.

Here is the graph of $s(t)$ from $t = 0$ to $t = 6.91$.



Here's a slightly different problem.

Chap 3.7 Exercise 50:

A car is traveling at 50 mi/h when the brakes are fully applied, producing a constant deceleration of 22 ft/s^2. What is the distance traveled before the car comes to a stop?

↓

How do we start?

Let $s(t)$ be the position of the car at time t, so $s'(t)$ is the velocity and $s''(t)$ is the acceleration at time t.

OK, what do we know?

We are given that $s'(0) = 50$ mi/h and $s''(t) = -22$ ft/s^2 for all $t > 0$.

Ok, what do we want?

We want to know where the car will stop. In other words we want to know $s(t)$ at the time t when the car stops. Well, when is this? The car stops when

$$s'(t) = 0.$$

OK, so first we should compute $s'(t)$. Since $s''(t) = -22$ is constant we have

$$s'(t) = \int s''(t)\,dt$$

$$= \int -22\,dt$$

$$= -22t + c$$

for some constant c. To compute c plug in $t=0$ to get

$$s'(0) = -22 \cdot 0 + c$$
$$50 = c.$$

But wait a minute. Are we using units of feet or miles. Probably we want feet. So we should convert 50 mi/hr into feet per second. Google says

$$50 \text{ mi/hr} = 73.33 \text{ ft/s}.$$

So we really have $c = 73.33$. Thus

$$s'(t) = -22t + 73.33$$

The car stops when $s'(t) = 0$.

$$-22t + 73.33 = 0.$$
$$73.33 = 22t$$
$$3.33 = t.$$

The car stops at $t = 3.33$ seconds.

Where does the car stop? At position $s(3.33)$. To compute this we first need to know $s(t)$. We have

$$s(t) = \int s'(t)\,dt$$

$$= \int (-22t + 73.33)\,dt$$

$$= -22 \cdot \frac{t^2}{2} + 73.33t + d$$

for some constant d. To compute d plug in $t = 0$ to get

$$s(0) = -22 \cdot 0 + 73.33 \cdot 0 + d$$
$$= d.$$

But what is $s(0)$? We don't know. The problem didn't tell us. It's the position of the car at time $t = 0$. I guess we can say whatever we want.

Let's just leave it as $s(0)$, so

$$s(t) = -11t^2 + 73.33t + s(0).$$

Finally we have

$$s(3.33) = -11(3.33)^2 + 73.33(3.33) + s(0)$$
$$= 122.22 + s(0) \text{ feet.}$$

We ~~don't~~ don't know what $s(0)$ is but we don't care. The car will stop 122.22 feet further down the road from where it started.

That seems reasonable.