7/13/15

Quiz 2 NOW (25 minutes).

Now we enough tricks to compute the derivative of pretty much any function you can write down, but some of them still need some practice.

Recall from Friday the Product Rule & Chain Rule.

The Product Rule: Given $f(x)$ & $g(x)$,

$$(f(x)\cdot g(x))' = f'(x)\cdot g(x) + f(x)\cdot g(x)$$

Example: Let $f(x) = x\cdot \sin(x)$. Then

$$\begin{aligned} f'(x) &= (x\cdot \sin(x))' \\ &= (x)'\cdot \sin(x) + x\cdot(\sin(x))' \\ &= 1\cdot \sin(x) + x\cdot \cos(x). \end{aligned}$$

☆ The Chain Rule: Given $f(x)$ & $g(x)$ we have

$$(f(g(x)))' = f'(g(x)) \cdot g'(x)$$

Which we could also phrase as

$$(f(\text{inside function}))'$$
$$= f'(\text{inside function}) \cdot (\text{inside function})'$$

Example: Let $f(x) = \sin(x^2+1)$. Then

$$f'(x) = (\sin(x^2+1))'$$
$$= \cos(x^2+1) \cdot (x^2+1)'$$
$$= \cos(x^2+1) \cdot (2x+0)$$
$$= 2x\cos(x^2+1).$$

The chain rule is easier to remember if we write it in Leibniz notation.

★ The Chain Rule in Leibniz Notation:

Let y be a function of u and let u be a function of x (so y is also a function of x). Then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

(That just <u>looks</u> <u>true</u> :))

Example: Let $y = \sin(u)$ & $u = x^2+1$, so $y = \sin(x^2+1)$. Then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \cos(u) \cdot (2x+0).$$

$$= 2x \cdot \cos(u)$$

$$= 2x \cdot \cos(x^2+1).$$

///

Same answer. ✓

By putting the Product & Chain Rules together, we can prove the Quotient Rule.

The Quotient Rule: Given $f(x)$ & $g(x)$,

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{g(x)^2}$$

Proof: Rewrite $f(x)/g(x)$ as $f(x) \cdot g(x)^{-1}$. Then use the Product & Chain Rules to obtain

$$\left(f(x) \cdot g(x)^{-1}\right) = f'(x) \cdot g(x)^{-1} + f(x) \cdot \left(g(x)^{-1}\right)'$$

$$= f'(x) \cdot g(x)^{-1} + f(x) \cdot (-1) g(x)^{-2} \cdot g'(x).$$

$$= \frac{f'(x)}{g(x)} - \frac{f(x)\, g'(x)}{g(x)^2}$$

$$= \frac{f'(x)\, g(x)}{g(x)^2} - \frac{f(x)\, g'(x)}{g(x)^2}$$

$$= \frac{f'(x)\, g(x) - f(x)\, g(x)}{g(x)^2}$$

Now let's practice :

Differentiate the following functions.

1. $f(x) = \sin\left(\cos\left(x^2+1\right)\right)$.

2. $g(u) = \left(u^2+u+1\right)^5$

3. $y = x / \sqrt{x^2-1}$.

4. $y = \sqrt{x} \cdot \cos\left(\sqrt{x}\right)$

1. Here we need to apply the chain rule twice

$$f'(x) = \left(\sin\left(\cos\left(x^2+1\right)\right)\right)'$$

$$= \cos\left(\cos\left(x^2+1\right)\right) \cdot \left(\cos\left(x^2+1\right)\right)'$$

$$= \cos\left(\cos\left(x^2+1\right)\right) \cdot \left(-\sin\left(x^2+1\right)\right) \cdot \left(x^2+1\right)'$$

$$= -\cos\left(\cos\left(x^2+1\right)\right) \cdot \sin\left(x^2+1\right) \cdot (2x).$$

$$= -2x \cos\left(\cos\left(x^2+1\right)\right) \cdot \sin\left(x^2+1\right).$$

2. We could do this by expanding $(u^2+u+1)^5$ first but it would be extremely tedious. Instead we use the chain rule.

$$g'(u) = \left((u^2+u+1)^5\right)'$$

$$= 5(u^2+u+1)^4 \cdot (u^2+u+1)'$$

$$= 5(u^2+u+1)^4 \cdot (2u+1).$$

3. $y = x/\sqrt{x^2-1}$.

We use the quotient rule and the chain rule to get

$$\frac{dy}{dx} = \frac{\sqrt{x^2-1}\cdot(x)' - x\cdot\left(\sqrt{x^2-1}\right)'}{\left(\sqrt{x^2-1}\right)^2}$$

$$= \frac{\sqrt{x^2-1}\cdot 1 - x\cdot\frac{1}{2}(x^2-1)^{-\frac{1}{2}}\cdot(2x).}{(x^2-1)}$$

$$= \frac{\sqrt{x^2-1} - x^2/\sqrt{x^2-1}}{x^2-1}$$

Actually there is lucky simplification here. Multiply top and bottom by $\sqrt{x^2-1}$ to get

$$f'(x) = \frac{\sqrt{x^2-1} - x^2/\sqrt{x^2-1}}{(x^2-1)} \cdot \frac{\sqrt{x^2-1}}{\sqrt{x^2-1}}$$

$$= \frac{(\cancel{x^2}-1) - \cancel{x^2}}{(x^2-1)\sqrt{x^2-1}}$$

$$= -1 \Big/ (x^2-1)^{3/2}.$$

4. $y = \sqrt{x} \cdot \cos(\sqrt{x})$.

Use the product and chain rules.

$$\frac{dy}{dx} = (\sqrt{x})' \cdot \cos(\sqrt{x}) + \sqrt{x}\,(\cos(\sqrt{x}))'$$

$$= \frac{1}{2} x^{-\frac{1}{2}} \cdot \cos(\sqrt{x}) + \sqrt{x}\left(-\sin(\sqrt{x})\right) \cdot \frac{1}{2} x^{-\frac{1}{2}}.$$

$$= \frac{1}{2\sqrt{x}} \cos(\sqrt{x}) - \frac{\cancel{\sqrt{x}}}{2\cancel{\sqrt{x}}} \sin(\sqrt{x}).$$

///

You will have more chance to practice computing derivatives on HW3. This is the foundation for further work so please make sure that you get enough practice.

Now that we can (in principle) compute the derivative of any function, we'll spend the week discussing applications.

To begin we'll discuss a "toy" problem that is very popular in Calculus books (see pg 175 of Stewart and pg. 75 of Spivak).

Problem: We want to build a cylindrical tin can to contain 1 ft^3 of soup. What is the minimum amount of tin we will need?

Stay tuned...

7/14/15

HW 3 is due Friday.

HW 2	Total	25
	Ave.	21.1
	Med.	22.5
	St. Dev.	4.1

Quiz 2	Total	10
	Ave.	6.6
	Med	8.5
	St. Dev	2.5

Still not as high as I would like.

You will practice computing derivatives on HW3. Please do extra practice problems if you still feel uncomfortable.

For the rest of the week we will discuss applications of the derivative. We'll begin with a popular "toy" problem.

<u>Problem</u> (pg 175 Stewart, pg. 75 Spivak):

We want to make a cylindrical tin can with capacity 1 ft^3. What is the <u>minimum</u> amount of tin we will need?

OK, a word problem like this requires some open-ended thinking...

What does a cylinder look like?

It has two parameters, the radius r and the height t.

The volume of a cylinder is

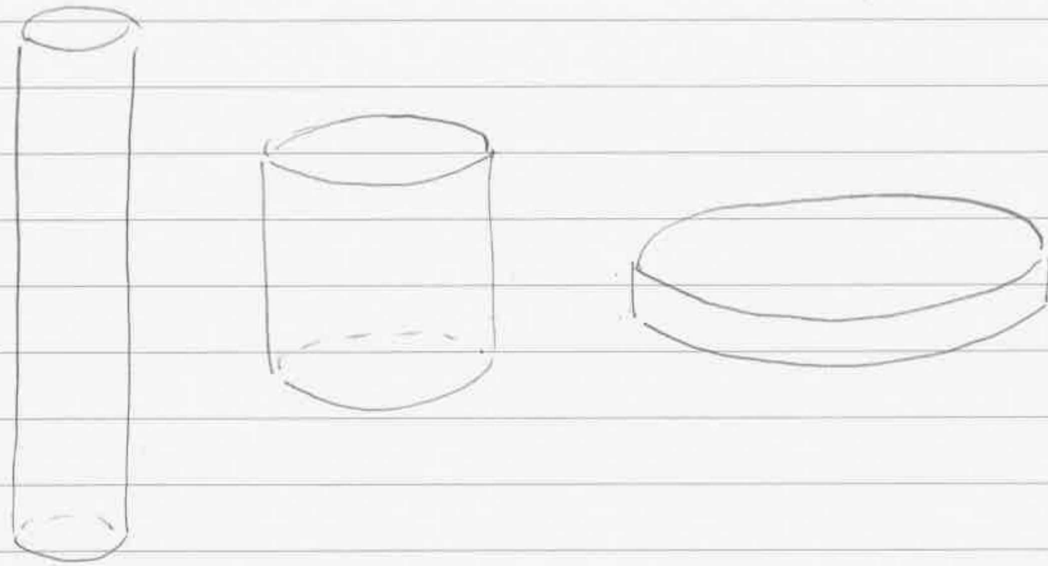
$$
\begin{aligned}
V &= (\text{area of base}) \times (\text{height}) \\
&= (\pi r^2) \cdot (h) \\
&= \pi r^2 h .
\end{aligned}
$$

For our can we will assume that

$$V = 1 \text{ ft}^3 \quad \text{is fixed.}$$

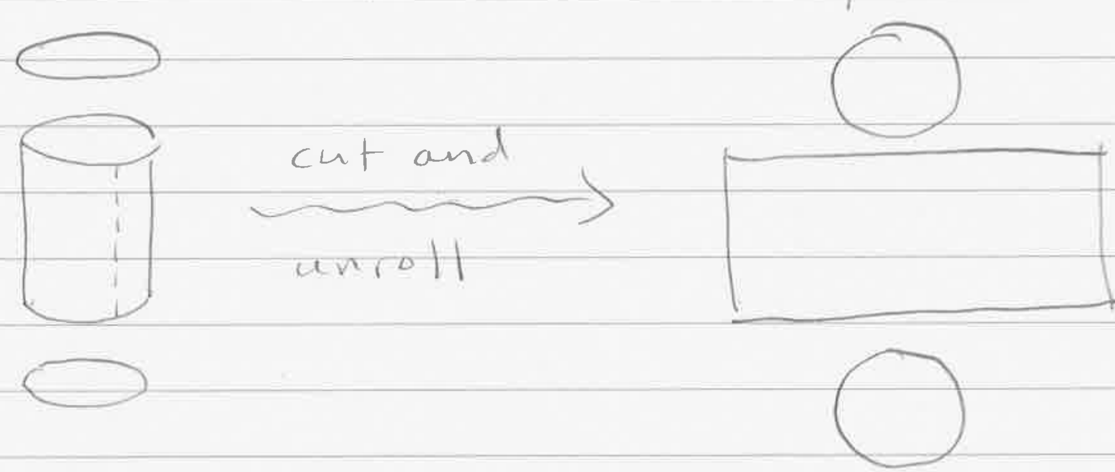
Still the values of r and h can change.

These three cans have the same volume $V=1$ but different values of r & h:

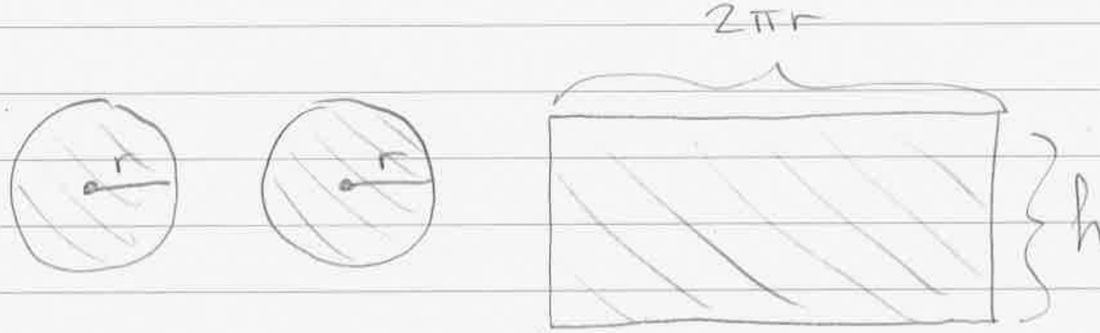


We want to find values of r & h that will minimize the amount of tin used.

The amount of tin is measured by the surface area of the can (at least approximately). To compute the surface area we can take the can apart:



We get two circles and a rectangle.



The total surface area is

$$\pi r^2 + \pi r^2 + 2\pi r \cdot h.$$

The area is a function of r & h,

$$\begin{aligned} A(r,h) &= \pi r^2 + \pi r^2 + 2\pi r h \\ &= 2\pi r^2 + 2\pi r h \\ &= 2\pi r (r+h). \end{aligned}$$

but since the volume is fixed,

$$V = \pi r^2 h = 1,$$

we can think of h as a function of r if we want: $h = V/\pi r^2 = 1/\pi r^2$.

Then area is a function of r alone:

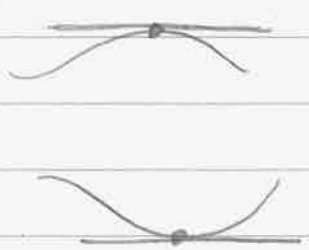
$$A(r) = 2\pi r(r+h)$$
$$= 2\pi r\left(r + \frac{1}{\pi r^2}\right)$$
$$= 2\pi r^2 + \frac{2}{r}.$$

Since area = tin (approximately), we want to find the value of r that makes the area $A(r)$ as small as possible.

How do we do that?

Recall when we were sketching the graphs of derivatives we noticed that:

if $f(x)$ has a local min or max at $x=a$ then $f'(a) = 0$.



local min or max means the tangent is horizontal (has slope zero).

So if want to find maxima and minima of $A(r)$ we should compute $A'(r)$ and look for r such that $A'(r) = 0$.

$$A(r) = 2\pi r^2 + \frac{2}{r}$$

$$= 2\pi r^2 + 2 \cdot r^{-1}$$

$$A'(r) = 2\pi \cdot 2r + 2(-1)r^{-2}$$

$$= 4\pi r - \frac{2}{r^2}.$$

Now set $A'(r) = 0$ to get

$$4\pi r - \frac{2}{r^2} = 0$$

$$4\pi r = 2/r^2$$

$$4\pi r^3 = 2$$

$$r^3 = 2/(4\pi)$$

$$r^3 = 1/2\pi$$

$$r = (1/2\pi)^{1/3}.$$

So if the area $A(r)$ has a minimum at all (which it might not) then this minimum must occur when

$$r = (1/2\pi)^{1/3} = 0.542 \text{ feet}$$
$$= 6.50 \text{ inches}$$

[We will discuss later how to show that this really is a minimum. For now we'll just believe it.]

When $r = (1/2\pi)^{1/3}$, the height of the can is

$$h = \frac{1}{\pi r^2} = \frac{1}{\pi (1/2\pi)^{2/3}}$$

$$= 1.08 \text{ feet}$$
$$= 13.00 \text{ inches}$$

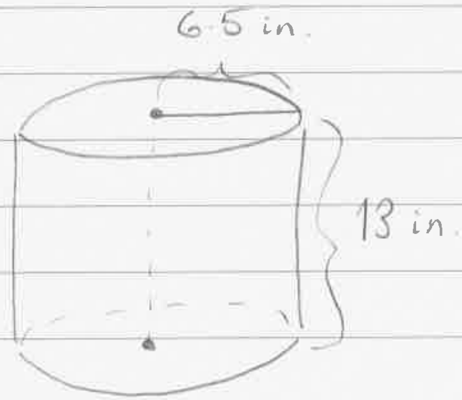
and the minimum amount of tin required is the minimum value of $A(r)$, i.e.,

$$A\left(\left(\tfrac{1}{2\pi}\right)^{1/3}\right) = 2\pi\left[\left(\tfrac{1}{2\pi}\right)^{1/3}\right]^2 + \frac{2}{\left(\tfrac{1}{2\pi}\right)^{1/3}}$$

$$= 5.54 \text{ feet}^2$$

In summary, to make a tin can with volume 1 ft^3, we need at least 5.54 ft^2 of tin. The minimum occurs when $r = 6.5$ in. and $h = 13.00$ in.

$V = 1 \ ft^3$
(fixed)



$A = 5.54 \ ft^2$
(minimum)

[Could we have done this without Calculus? Probably not.]

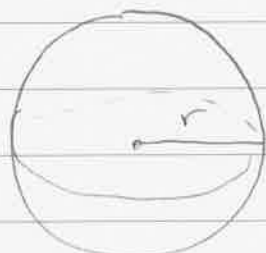
For comparison, a tin cube of volume 1 ft^3 has surface area 6 ft^2, so a cylinder is more efficient at containing volume than a cube.

$$5.54 < 6$$

How about a sphere?

Exercise: Suppose a tin sphere has volume $1\ ft^3$. In this case, what is its surface area?

Consider a sphere of radius r. Look in the back of the book to find its volume and surface area:



$$V = \frac{4}{3}\pi r^3$$

$$A = 4\pi r^2.$$

If $V = 1$ we compute the radius.

$$\frac{4}{3}\pi r^3 = 1$$

$$r^3 = \frac{3}{4\pi}$$

$$r = (3/4\pi)^{1/3}.$$

Then we compute the surface area.

$$A = 4\pi r^2$$
$$= 4\pi\left(\left(3/4\pi\right)^{1/3}\right)^2$$
$$= 4\pi\left(3/4\pi\right)^{2/3}$$
$$= 4.84 \text{ ft}^2$$

We conclude that a sphere is more efficient than a cylinder.

Summary: Amount of surface area (tin) needed to contain 1 unit of volume (soup).

$$\underset{\text{sphere}}{4.84} < \underset{\text{cylinder}}{5.54} < \underset{\text{cube}}{6}$$

[Since you don't often see spherical soup cans, there must be some other considerations involved. That's why we call this a "toy" problem.]

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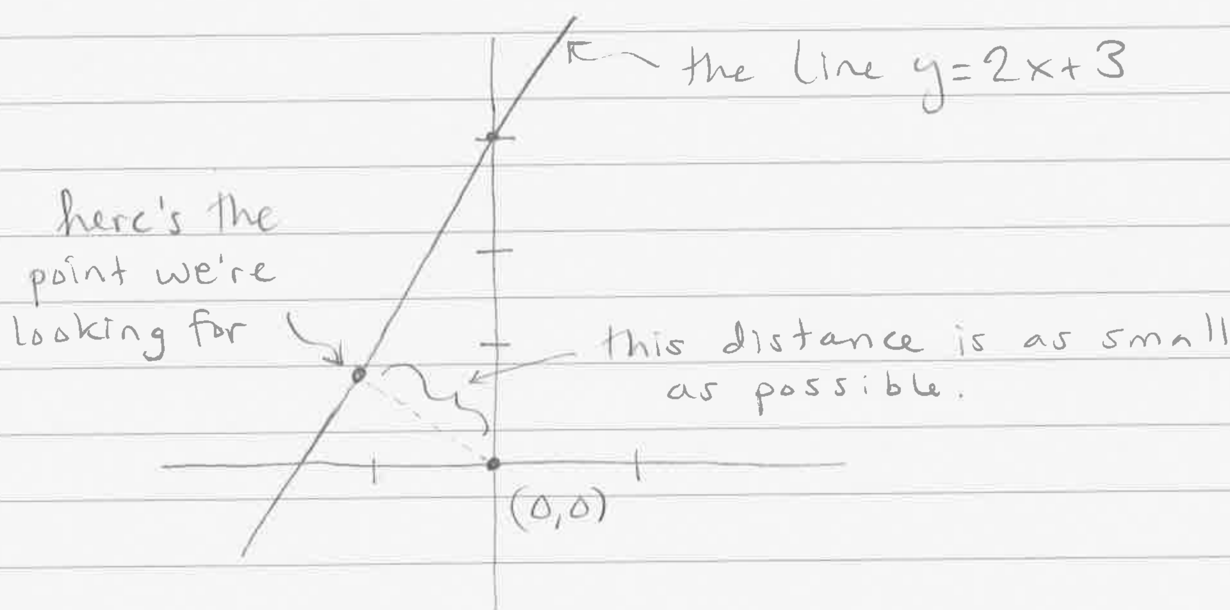
Recitation:

Find the point on the line $y = 2x + 3$ that is closest to the origin.

First draw a picture to help the imagination.



OK, so how do we find the coordinates of this point?

We want to find the point (x,y) on the line such that the distance between $(0,0)$ and (x,y) is as small as possible.

The distance from $(0,0)$ to (x,y) is

$$\sqrt{(x-0)+(y-0)^2} = \sqrt{x^2+y^2}.$$

Let $D(x,y) = \sqrt{x^2+y^2}$.

Obviously $D(x,y)$ is minimized when $x=0$ and $y=0$ but that's not interesting. We are only interested in the value of $D(x,y)$ when the point (x,y) is on the line $y=2x+3$.

In this case we can express $D(x,y)$ as a function of x alone:

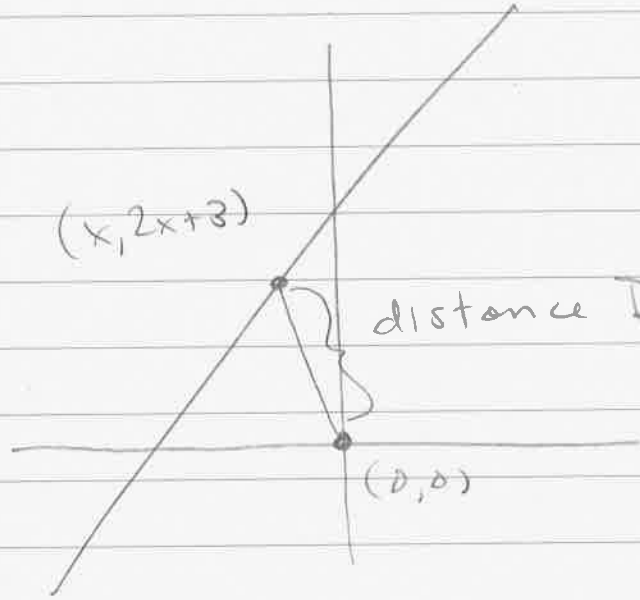
$$D(x) = \sqrt{x^2+y^2}$$

$$= \sqrt{x^2+(2x+3)^2}$$

$$= \sqrt{x^2+4x^2+12x+9}$$

$$= \sqrt{5x^2+12x+9}.$$

Here's the picture:



distance $D(x) = \sqrt{5x^2 + 12x + 9}$

We want to find x (and hence $(x, 2x+3)$) such that $D(x)$ is minimized.

When $D(x)$ is minimized we must have $D'(x) = 0$. So we compute

$$D'(x) = \left(\sqrt{5x^2+12x+9}\right)'$$

$$= \left(\sqrt{\text{something}}\right)'$$

$$= \frac{1}{2\sqrt{\text{something}}} \cdot (\text{something})'$$

$$= \frac{1}{2\sqrt{5x^2+12x+9}} \cdot (10x+12)$$

and then solve

$$D'(x) = 0$$

$$\frac{10x + 12}{2\sqrt{5x^2 + 12x + 9}} = 0$$

$$10x + 12 = 0$$

$$x = -12/10 = -6/5 = -1.2$$

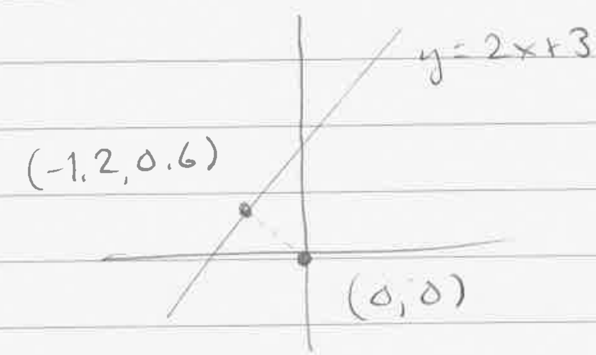
The point we are looking for is

$$(x, 2x+3) = \left(-\frac{6}{5}, 2\left(-\frac{6}{5}\right) + 3\right)$$

$$= \left(-\frac{6}{5}, -\frac{12}{5} + 3\right)$$

$$= \left(-\frac{6}{5}, \frac{3}{5}\right)$$

$$= (-1.2, 0.6).$$



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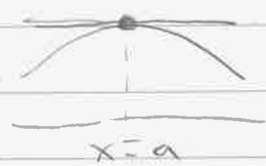
7/15/15.

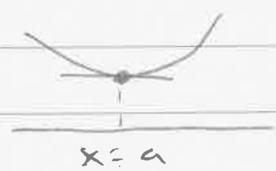
HW 3 due Friday.

Yesterday we used derivatives to minimize and maximize functions.

Idea: If $f(x)$ has a local minimum or local maximum at $x=a$ then the tangent line at $(a, f(a))$ is horizontal, so it has slope 0 (i.e. $f'(a)=0$).



$\Longrightarrow \quad f'(a)=0$



$\Longrightarrow \quad f'(a)=0$.

But how can we tell the difference between a max and a min without graphing it on our calculator/computer?

Yesterday were were a bit sloppy about this.

$\downarrow$

Example: Find r to minimize

$$A(r) = 2\pi r^2 + \frac{2}{r}$$

Compute $A'(r)$:

$$A'(r) = 4\pi r - \frac{2}{r^2}$$

and then solve $A'(r) = 0$:

$$4\pi r - \frac{2}{r^2} = 0$$

$$4\pi r = \frac{2}{r^2}$$

$$4\pi r^3 = 2$$

$$r^3 = \frac{2}{4\pi} = \frac{1}{2\pi}$$

$$r = \left(1/2\pi\right)^{1/3} = 0.54.$$

We conclude that <u>if</u> $A(r)$ has a minimum then this minimum must occur when $r = 0.54$. But maybe $A(r)$ doesn't have a minimum. Maybe it actually has a <u>maximum</u> at $r = 0.54$ ...

How can we tell?

The Key is to look at the 2nd derivative.

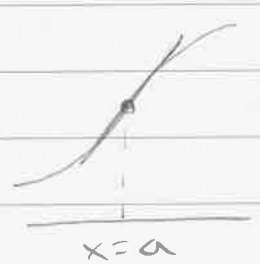
$$A''(r) = \left(4\pi r - \frac{2}{r^2}\right)'$$

$$= 4\pi - 2(-2)r^{-3}$$

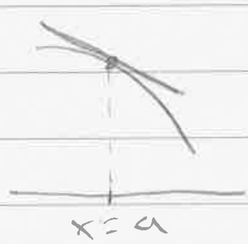
$$= 4\pi + \frac{4}{r^3}.$$

OK, what does this tell us?

Recall: If $f'(a) > 0$ then $f(x)$ is increasing at $x = a$ and if $f'(a) < 0$ then $f(x)$ is decreasing at $x = a$.



$\iff f'(a) > 0$

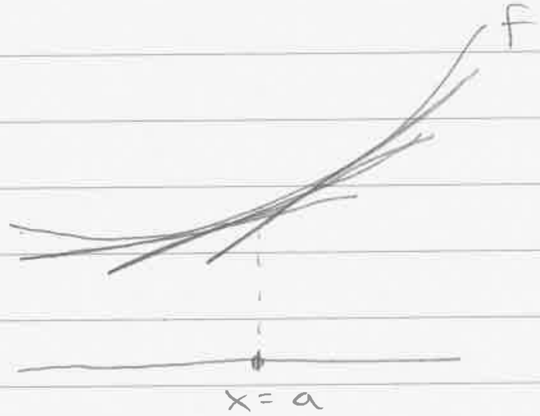


$\iff f'(a) < 0$

By the same reasoning, if $f''(a) > 0$ then $f'(x)$ is increasing at $x = a$ and if $f''(a) < 0$ then $f'(x)$ is increasing at $x = a$.

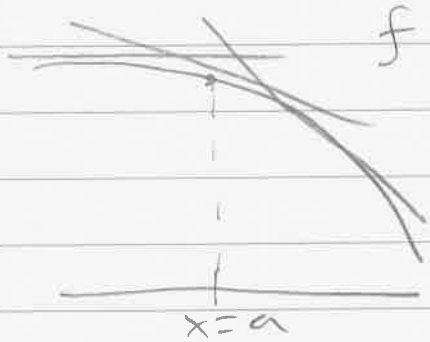
What does this look like?

Suppose $f''(a) > 0$, so $f'(x)$ is increasing at $x = a$. This means that the slope of the tangent to $f(x)$ is increasing:



slope is increasing near $x = a$.

In this case we say that the graph of $f(x)$ is <u>concave</u> <u>up</u> near $x = a$. Similarly, when $f''(a) < 0$ the graph of $f(x)$ will be <u>concave</u> <u>down</u> near $x = a$:



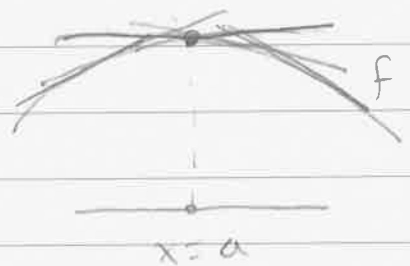
$f''(a) < 0$ means slope is decreasing near $x = a$

Hey, this is exactly what we need to distinguish maxima and minima.

☆ The Second Derivative Test (pg. 163):

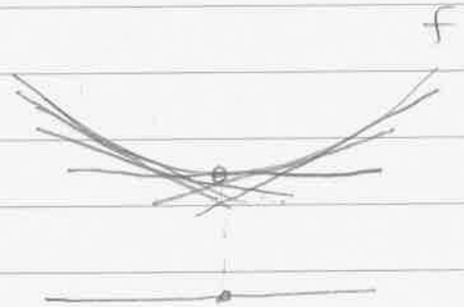
Let $f(x)$ be a function.

- If $f'(a) = 0$ and $f''(a) < 0$ then $f(x)$ has a local maximum at $x = a$:



$f'(a) = 0$ (horiz. tangent)
$f''(a) < 0$ (concave down)

- If $f'(a) = 0$ and $f''(a) > 0$ then $f(x)$ has a local minimum at $x = a$:



$f'(a) = 0$ (horiz. tangent)
$f''(a) > 0$ (concave up)

That's useful!

Let's use it. Recall that

$$A'(0.54) = 0$$

Is this a max or min? We computed

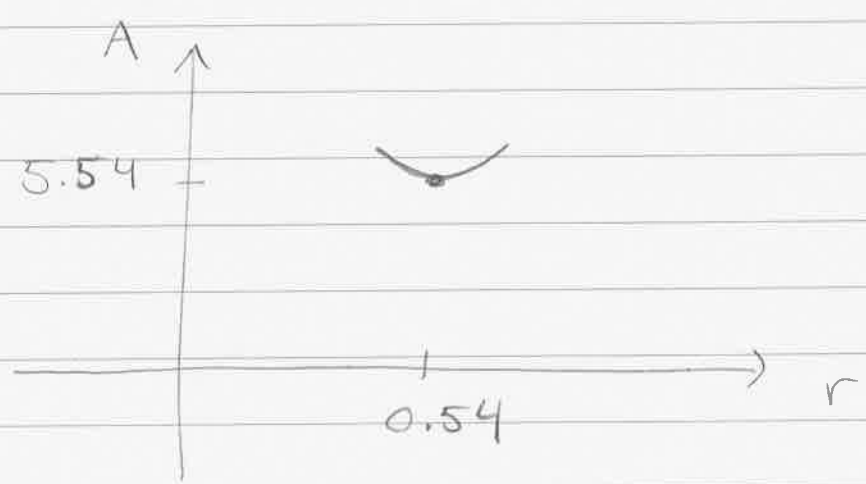
$$A''(r) = 4\pi + \frac{4}{r^3}.$$

Now plug in $r = 0.54$ to get

$$A''(0.54) = 4\pi + 4/(0.54)^3$$

$$= 37.7 > 0$$

By the 2nd Derivative Test we conclude that $A(r)$ has a local MINIMUM at $r = 0.54$., i.e., the graph of $A(r)$ near $r = 0.54$ looks like



Example: Chap 3.3 Exercise 1.

Let $f(x) = 2x^3 + 3x^2 - 36x$.

(a) Determine when f is increasing/decreasing.
(b) Find the local maxima/minima.
(c) Determine when f is concave up/down.
(d) Sketch the graph.

(a) $$\begin{aligned} f'(x) &= 2\cdot 3x^2 + 3\cdot 2x - 36 \\ &= 6x^2 + 6x - 36. \\ &= 6(x^2 + x - 6) \\ &= 6(x-2)(x+3). \end{aligned}$$ LUCKY ☺

Solve $f'(x) = 0$.

$$6(x-2)(x+3) = 0$$
$$(x-2)(x+3) = 0$$

$$\Longrightarrow \underbrace{x = -3 \text{ or } x = +2}_{\text{called the "critical numbers" of } f.}$$

In between the critical numbers we must have either $f' > 0$ or $f' < 0$.

• when $x < -3$ we have $(x-2) < 0$ & $(x-3) < 0$

so $f'(x) = 6\underset{\ominus}{(x-2)}\underset{\ominus}{(x+3)} > 0$ INCREASING

• when $-3 < x < 2$ we have $(x-2) < 0$ & $(x+3) > 0$

so $f'(x) = 6\underset{\ominus}{(x-2)}\underset{\oplus}{(x+3)} < 0$ DECREASING

• when $2 < x$ we have $(x-2) > 0$ & $(x+3) > 0$

so $f'(x) = 6\underset{\oplus}{(x-2)}\underset{\oplus}{(x+3)} > 0$. INCREASING.

(b) To check if $x = -3$ & $x = 2$ are minima or maxima, compute $f''(x)$.

$$f'(x) = 6(x-2)(x+3)$$
$$= 6x^2 + 6x - 36$$

$$f''(x) = 6 \cdot 2x + 6 - 0$$
$$= 12x + 6.$$

• $f'(-3) = 0$, $f''(-3) = 12(-3) + 6 = -30 < 0$

so $x = -3$ is local MAXIMUM.

- $f'(2) = 0$, $f''(2) = 12(2) + 6 = 30 > 0$

  so $x = 2$ is local MINIMUM.

(c) To check concavity determine when $f''(x)$ is > 0 & < 0.

- Concave up when

$$\begin{aligned} f''(x) &> 0 \\ 12x + 6 &> 0 \\ 12x &> -6 \\ x &> -1/2 \end{aligned}$$

- Concave down when

$$\begin{aligned} f''(x) &< 0 \\ 12x + 6 &< 0 \\ 12x &< -6 \\ x &< -1/2. \end{aligned}$$

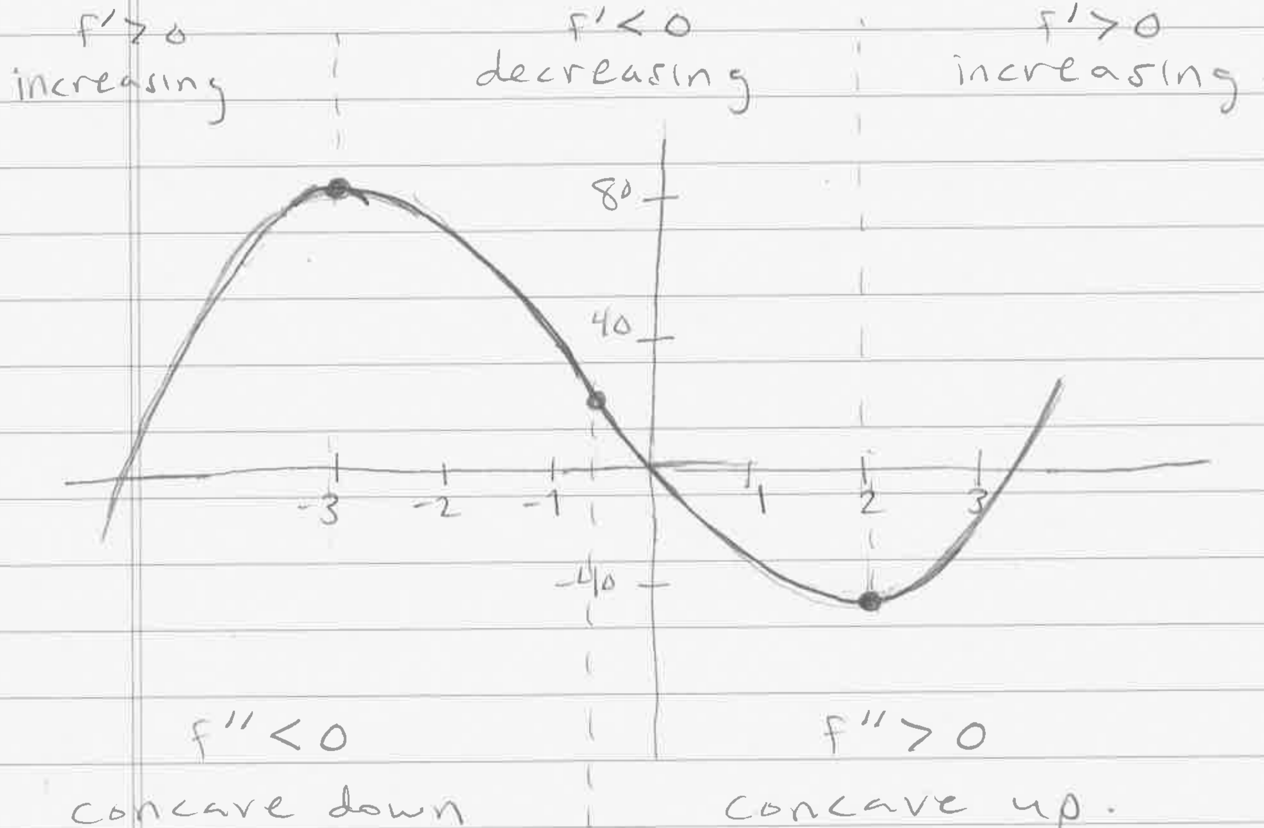
- Note that $f''(-1/2) = 0$. This means the graph of $f(x)$ has an "inflection point" at $x = -1/2$.

(d) Sketch the graph.

Local max at $(-3, f(-3)) = (-3, 81)$.

Local min at $(2, f(2)) = (2, -44)$

Inflection point at $\left(-\frac{1}{2}, f\left(-\frac{1}{2}\right)\right) = \left(-\frac{1}{2}, 18.5\right)$



[Note that we never found the zeros of f(x). Exercise: Do this and check that it fits with the sketch.]

Recitation: Let $f(x) = x/(x^2+1)$.

Determine when $f(x)$, $f'(x)$, $f''(x)$ are $=0$, <0, >0, and use this information to sketch the graph.

OK, let's do this.

- $f(x)$.

First note that $(x^2+1) > 0$ for all x, so we have

$$f(x) < 0 \quad \text{when} \quad x < 0$$
$$f(x) = 0 \quad \text{when} \quad x = 0$$
$$f(x) > 0 \quad \text{when} \quad x > 0.$$

- $f'(x)$.

Using the Quotient Rule we compute

$$f'(x) = \frac{(x^2+1)(x)' - x(x^2+1)'}{(x^2+1)^2}$$

$$= \frac{(x^2+1) - x(2x)}{(x^2+1)^2}$$

$$= \frac{x^2+1-2x^2}{(x^2+1)^2}$$

$$= \frac{1-x^2}{(x^2+1)^2} = \frac{(1-x)(1+x)}{(x^2+1)^2}.$$

Since $(x^2+1)^2 >$ for all x we have

$f'(x) = 0$ when $\underbrace{x = -1 \text{ or } x = +1}_{\text{the "critical numbers"}}$.

What happens in-between?

If $x < -1$ then $f'(x) < 0$ (f DECREASING)
If $-1 < x < +1$ then $f'(x) > 0$ (f INCREASING)
If $+1 < x$ then $f'(x) < 0$ (f DECREASING)

- $f''(x)$.

Using the Quotient Rule again gives

$$f''(x) = \frac{(x^2+1)^2 \cdot (1-x^2)' - (1-x^2)\left[(x^2+1)^2\right]'}{\left[(x^2+1)^2\right]^2}$$

$$= \frac{(x^2+1)^2(-2x) - (1-x^2)\cdot 2(x^2+1)^1\cdot(2x)}{(x^2+1)^4}$$

$$= \frac{(x^2+1)\left[(x^2+1)(-2x) - 4x(1-x^2)\right]}{(x^2+1)^4}$$

$$= \frac{1}{(x^2+1)^3}\left[-\cancel{2x^3} - \cancel{2x} - \cancel{4x} + \cancel{4x^3}\right]$$

$$= \frac{1}{(x^2+1)^3}\left[2x^3 - 6x\right]$$

$$= 2x(x^2-3) \big/ (x^2+1)^3$$

$$= 2x(x-\sqrt{3})(x+\sqrt{3}) \big/ (x^2+1)^3 .$$

[Yes, that was a big calculation. Use a calculator or WolframAlpha to check your work.]

Since $(x^2+1)^3 > 0$ for all x, we have

$f''(x) = 0$ when $x = 0$, $x = -\sqrt{3}$, or $x = +\sqrt{3}$.

What happens in-between?

$x < -\sqrt{3} \implies f''(x) < 0$ (f concave DOWN)
$-\sqrt{3} < x < 0 \implies f''(x) > 0$ (f concave UP)
$0 < x < +\sqrt{3} \implies f''(x) < 0$ (f concave DOWN)
$+\sqrt{3} < x \implies f''(x) > 0$ (f concave UP).

Also we check the critical numbers:

$f''(-1) > 0 \implies f$ has Local MIN at $x = -1$
$f''(+1) < 0 \implies f$ has local MAX at $x = +1$.

Finally let's sketch the graph of $f(x)$.

Local min at $(-1, f(-1)) = (-1, -1/2)$
Local max at $(+1, f(+1)) = (1, 1/2)$.

Inflection points at

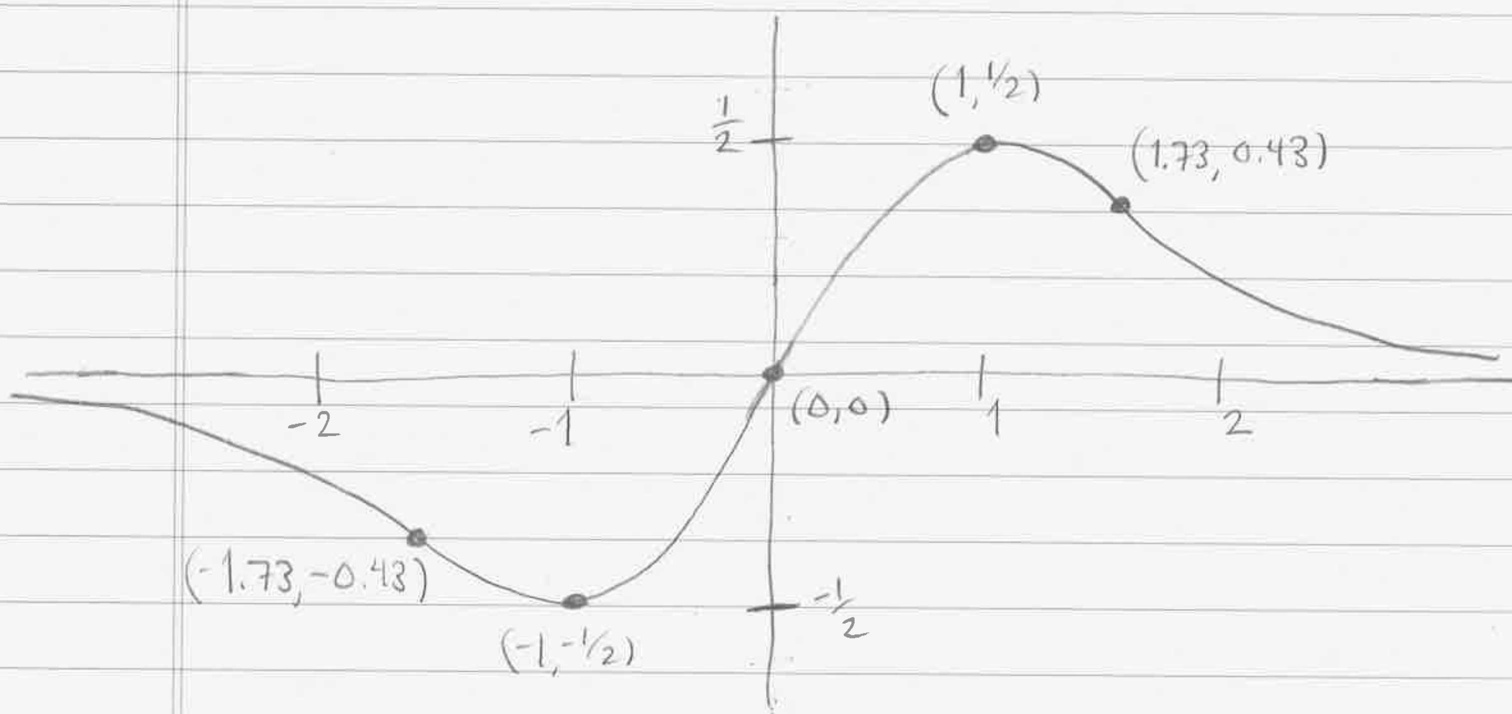
$(-\sqrt{3}, f(-\sqrt{3}))$, $(0, f(0))$, $(+\sqrt{3}, f(+\sqrt{3}))$

$(-1.73, -0.43)$, $(0, 0)$, $(1.73, 0.43)$.

We might also want to check that

$$\lim_{x \to \infty} f(x) = \lim_{x \to -\infty} f(x) = 0.$$

Here is the graph of $f(x)$:



Check that the graph agrees with all the properties we found.

[HW3 problem 3.3.34 is similar to this one.]

7/16/15

HW 3 due tomorrow.
Office Hours today 1-2pm Mem. 216.

We are discussing applications of the derivative. So far we have seen

3.3 & 3.4 Curve Sketching
3.5 Optimization Problems.

Today we will discuss the idea of "Linearization" (Chapter 2.8).

In mathematics we often want to solve equations. The easiest kinds of equations to solve are the <u>linear equations</u>.

Example: Solve $ax + b = c$ for x.

$$ax + b = c$$
$$ax = c - b$$
$$x = (c-b)/a.$$

DONE :)

Humans are so good at solving linear equations & systems of linear equations, there is a whole branch of mathematics devoted to it called

"Linear Algebra" (MTH 210, etc...)

Unfortunately, most equations that we want to solve are <u>not</u> linear.

The main strategy for solving a nonlinear function

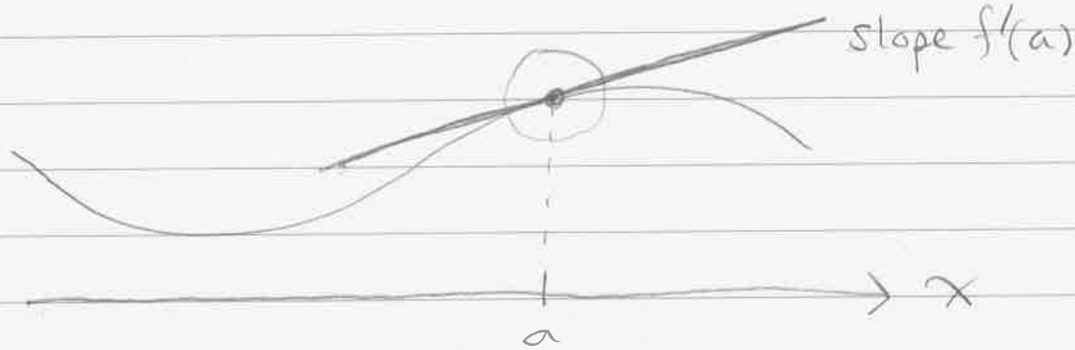
$$f(x) = c$$

is to "linearize" it, i.e., to turn it into a linear function.

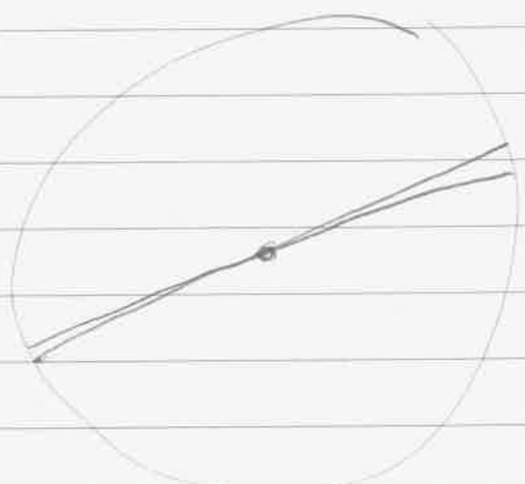
OK, what does that mean?

Recall the geometric interpretation of the derivative:

Given function $f(x)$, the number $f'(a)$ is the slope of the tangent line to the graph at point $(a, f(a))$.



If we zoom into the point $(a, f(a))$ note the tangent line is very close to the graph:



So if we are only interested in x near a, it might be reasonable to replace the function $f(x)$ with the equation of its tangent line.

What is the equation of the tangent?

It contains point $(a, f(a))$ and has slope $f'(a)$ so the equation is

$$\frac{y - f(a)}{x - a} = f'(a)$$

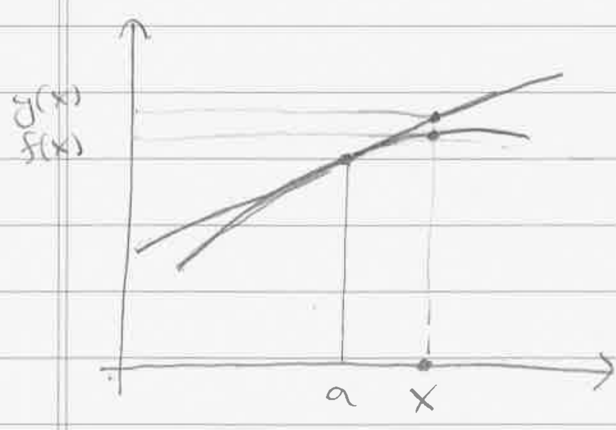
$$y - f(a) = f'(a)(x-a)$$

$$y = f(a) + f'(a)(x-a).$$

This y is a <u>linear</u> function of x:

$$y(x) = f(a) + f'(a)(x-a)$$

and for x near a the value $y(x)$ is very close to the value $f(x)$ (because the tangent line is close to the curve)



For $x \approx a$ we have $f(x) \approx y(x)$.

$$\boxed{f(x) \approx f(a) + f'(a)(x-a)}$$

This is called the linearization of $f(x)$ near $x = a$.

Let's try it out.

Example: Find the linearization of the (nonlinear) function $f(x) = \sqrt{x}$ near $x = 4$.

Compute $f'(x) = \left(x^{1/2}\right)' = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$.

Then for $x \approx 4$ we have

$$f(x) \approx f(4) + f'(4)(x-4)$$

$$= \sqrt{4} + \frac{1}{2\sqrt{4}}(x-4)$$

$$= 2 + \frac{1}{4}(x-4)$$

$$= 2 + x/4 - 1$$

$$= 1 + x/4.$$

$$\boxed{\sqrt{x} \approx 1 + \frac{x}{4} \quad \text{when } x \approx 4}$$

Now let's test it.

$$\sqrt{4.1} \approx 1 + \frac{4.1}{4} = 1 + 1.025$$
$$= 2.025.$$

My computer gives the exact value

$$\sqrt{4.1} = 2.0248\cdots$$

which is pretty close to 2.025.

As x gets farther away from 4 the approximation gets worse. E.g.,

$$\sqrt{5} \approx 1 + \frac{5}{4} = 2.25$$

but my computer says $\sqrt{5} = 2.236\cdots$.

So we have to be careful.

Practice: Estimate $1/4.002$ using a linear approximation.

Note that 4.002 is close to 4 so we should linearize the function $f(x) = 1/x$ at $x = 4$.

First compute $f'(x) = (x^{-1})' = (-1)x^{-2} = -\frac{1}{x^2}$.

Then $f(x) \approx f(4) + f'(4)(x-4)$

$$= \frac{1}{4} - \frac{1}{4^2}(x-4)$$

$$= \frac{1}{4} - \frac{1}{16}(x-4)$$

$$= \frac{1}{4} - \frac{x}{16} + \frac{4}{16}$$

$$= \frac{1}{2} - \frac{x}{16}.$$

$$\boxed{\frac{1}{x} \approx \frac{1}{2} - \frac{x}{16} \quad \text{when } x \approx 4}$$

So $\frac{1}{4.002} \approx \frac{1}{2} - \frac{4.002}{16}$

$$= 0.5 - 0.250125$$

$$= 0.249875$$

Is that a good approximation?

Check: $\dfrac{1}{4.002} = 0.2498750\underline{625}$.

Yes, it's very good because 4.002 is very close to 4.

The idea of linearization leads in two directions.

1. Note that the approximation

$$f(x) \approx f(a) + f'(a)(x-a)$$

gets worse as x gets farther away from a. We can sometimes fix this by using a higher degree approximation. For example, the 2nd <u>degree</u> <u>approximation</u> of $f(x)$ near $x=a$ is

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2.$$

This is more accurate than the linear approximation.

Continuing in this way leads to the "Taylor series expansion" of $f(x)$ near $x=a$.

Your calculator uses Taylor series, but this is the subject of Chap 8.7 so we can't talk about it today. :(

2. Let's try to express the linearization in Leibniz notation.

Recall that $f(x) \approx f(a) + f'(a)(x-a)$ when x is near a.

Instead of saying that x is near a, let's write $x = a + dx$ where dx is a very small (infinitesimal) change in x. This is the Leibniz style. Then

$$f(a+dx) \approx f(a) + f'(a)(\cancel{a} + dx - \cancel{a})$$

$$f(a+dx) \approx f(a) + dx \cdot f'(a).$$

But $f'(a)$ is not in Leibniz notation. Let's fix that.

Recall that f' means df/dx. So

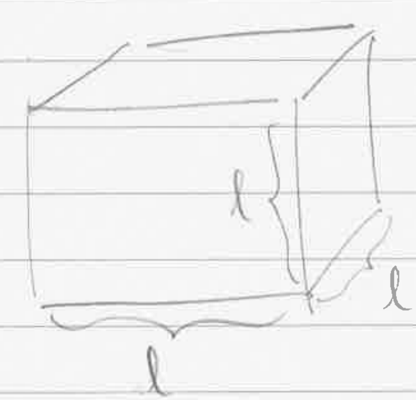
$$f(a+dx) \approx f(a) + dx \cdot f'(a)$$

$$= f(a) + \cancel{dx} \cdot \frac{df}{\cancel{dx}}$$

$$\boxed{f(a+dx) \approx f(a) + df}$$

This looks like magic so let's do an example. This idea is often used in science to compute experimental errors.

Example: We want to measure the surface area of a cube, so we get out our ruler and measure the length of an edge.



We find that

$$l = 30 \pm 0.1 \text{ cm}.$$

We compute the area and volume.

$$A = 6 \cdot \ell^2 = 6 \cdot (30)^2 = 5400 \text{ cm}^2$$

$$V = \ell^3 = (30)^3 = 27000 \text{ cm}^3.$$

But what is the error in A and V?

It's actually pretty tedious to carry the error through all the calculations, so scientists tend to use linearization to find the approximate error.

We have $d\ell = 0.1$ cm, so

$$A(30 + d\ell) \approx A(30) + dA.$$

What is dA? We know this:

$$A = 6 \cdot \ell^2$$

$$\frac{dA}{d\ell} = 6 \cdot 2\ell = 12\ell.$$

$$\implies dA = 12\ell \cdot d\ell$$

When $l = 30$ and $dl = 0.1$ we get

$$dA = 12l \cdot dl$$
$$= 12(30)(0.1) = 36 \text{ cm}^2.$$

In other words

$$A(30 + 0.1) \approx A(30) + 36.$$
$$= 5400 + 36 \text{ cm}.$$

Summary: If the error in l is ± 0.1 cm then the error in A is $\approx \pm 36 \text{ cm}^2$.

Now let's compute the error in the volume.

$$V = l^3$$

$$dV/dl = 3l^2$$

$$dV = 3l^2 \cdot dl.$$

When $l = 30$ and $dl = 0.1$ we get

$$dV = 3(30)^2(0.1) = 270 \text{ cm}^3.$$

WARNING: This is funny stuff. We discussed before how the dy and dx in the derivative dy/dx are not really numbers:

$\frac{dy}{dx}$ is not "dy divided by dx"

Nevertheless, it is sometimes useful and convenient to pretend that dx and dy are actual numbers. When we do this we call them differentials.

It's mostly harmless and (with a lot of work) it can be put on sound mathematical footing, so don't worry too much about it.

[... maybe I should have called it a Remark instead of a WARNING.]

Practice: Chap 2.8 Exercise 23.

The circumference of a sphere is measured to be 84 cm ± 0.5 cm

Compute the surface area & volume together with their errors.

Let c stand for circumference. Then

$$c = 50 \quad \& \quad dc = 0.5$$

The formulas for area & volume are

$$A = 4\pi r^2$$

$$V = \frac{4}{3}\pi r^3$$

where r is the radius of the sphere. The relation between circumference and radius is

$$\begin{cases} c = 2\pi r \\ r = c/2\pi. \end{cases}$$

Hence $A = 4\pi\left(\frac{c}{2\pi}\right)^2 = c^2/\pi$

$$V = \frac{4}{3}\pi\left(\frac{c}{2\pi}\right)^3 = c^3/(6\pi^2)$$

as functions of c.

Now we compute the differentials.

$$\frac{dA}{dc} = \frac{d}{dc}\left(\frac{c^2}{\pi}\right) = \frac{2c}{\pi}$$

$$\Longrightarrow \quad dA = \frac{2c}{\pi} \cdot dc$$

$$= \frac{2(50)(0.5)}{\pi} = 15.91 \text{ cm}^2$$

$$\frac{dV}{dc} = \frac{d}{dc}\left(\frac{c^3}{6\pi^2}\right) = \frac{3c^2}{6\pi^2} = \frac{c^2}{2\pi^2}$$

$$\Longrightarrow \quad dV = \frac{c^2}{2\pi^2} \cdot dc$$

$$= \frac{(50)^2(0.5)}{2\pi^2} = 63.33 \text{ cm}^3.$$

Now we compute A & V.

$$A = c^2/\pi = (50)^2/\pi = 795.77 \text{ cm}^2.$$

$$A = 795.77 \text{ cm}^2 \pm 15.91 \text{ cm}^2.$$

$$V = c^3/(6\pi^2) = (50)^3/(6\pi^2) = 2110.86 \text{ cm}^3.$$

$$V = 2110.86 \text{ cm}^3 \pm 63.33 \text{ cm}^3.$$

Sometimes people prefer to discuss the relative errors. These are

$$\frac{dA}{A} = \frac{15.91}{795.77} = 0.020 = 2.0\%$$

$$\frac{dV}{V} = \frac{63.33}{2110.86} = 0.030 = 3.0\%$$

The relative error in c is

$$\frac{dc}{c} = \frac{0.5}{50} = 0.010 = 1.0\%$$

[Note that the relative error got bigger when we calculated the surface area & volume.]