Math 161 **Summer 2015**
Homework 4 Solutions **Drew Armstrong**

Book Problems:

- Chap 3.7 Exercises 2, 4, 14
- Chap 4.1 Exercises 6
- Chap 4.2 Exercises 30, 38, 42
- Chap 4.3 Exercises 2, 6, 10, 14
- Chap 4.4 Exercises 6, 10

Solutions:

3.7.2. Find the most general antiderivative of $f(x) = 8x^9 - 3x^6 + 12x^3$.

Recall that $\int x^p\,dx = \frac{1}{p+1}x^{p+1}$ for all $p \neq -1$. Thus we have

$$\begin{aligned}
\int f(x)\,dx &= \int (8x^9 - 3x^6 + 12x^3)\,dx \\
&= 8\int x^9\,dx - 3\int x^6\,dx + 12\int x^3\,dx \\
&= 8\frac{1}{10}x^{10} - 3\frac{1}{7}x^7 + 12\frac{1}{4}x^4 + C,
\end{aligned}$$

where C is an arbitrary constant.

3.7.4. Find the most general antiderivative of $f(x) = \sqrt[3]{x^2} + x\sqrt{x}$.

First we write $f(x) = (x^2)^{1/3} + x^1 \cdot x^{1/2} = x^{2/3} + x^{3/2}$. Then we have

$$\begin{aligned}
\int f(x)\,dx &= \int (x^{2/3} + x^{3/2})\,dx \\
&= \int x^{2/3}\,dx + \int x^{3/2}\,dx \\
&= \frac{1}{5/3}x^{5/3} + \frac{1}{5/2}x^{5/2} + C \\
&= \frac{3}{5}x^{5/3} + \frac{2}{5}x^{5/2} + C,
\end{aligned}$$

where C is an arbitrary constant.

3.7.14. Find the most general antiderivative of $f(\theta) = 6\,\theta^2 - 7\sec^2\theta$.

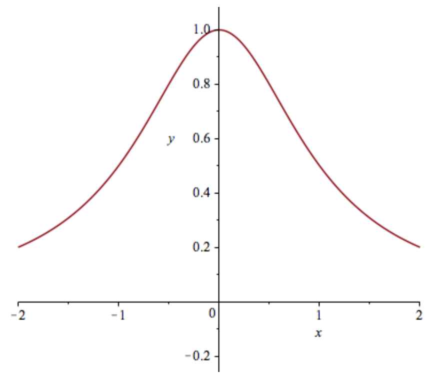
First recall that $\frac{d}{d\theta}\tan\theta = \sec^2\theta$. Thus we have

$$\begin{aligned}
\int f(\theta)\,d\theta &= \int (6\,\theta^2 - 7\sec^2\theta)\,d\theta \\
&= 6\int \theta^2\,d\theta - 7\int \sec^2\theta\,d\theta \\
&= 6\frac{1}{3}\theta^3 - 7\tan\theta + C,
\end{aligned}$$

where C is an arbitrary constant.

4.1.6. Graph the function $f(x) = 1/(1+x^2)$ for $-2 \le x \le 2$. Estimate the area under the graph by using four rectangles with left endoints, right entpoints, and midpoints. Then do the same with eight rectangles.

Here's the graph of $f(x)$ from $x = -2$ to $x = 2$:



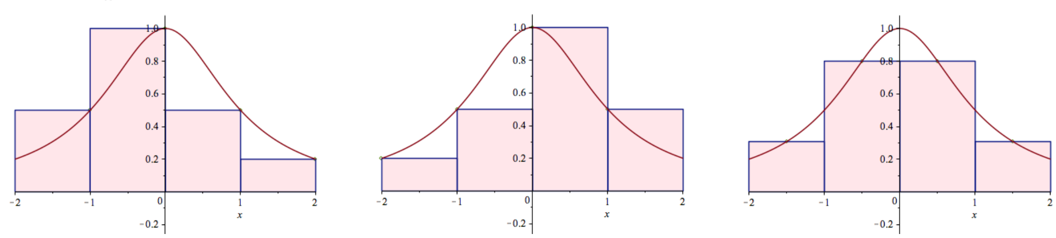
To approximate the area with four rectangles we let $n = 4$ so that $\Delta x = (2-(-2))/4 = 1$ and $x_i = -2 + i \cdot \Delta x = -2 + i$. The approximations using right hand endpoints, left hand endpoints, and midpoints are

$$R_4 = \sum_{i=1}^{n} f(x_i) \cdot \Delta x = f(-1) + f(0) + f(1) + f(2) = 2.2$$

$$L_4 = \sum_{i=1}^{n} f(x_{i-1}) \cdot \Delta x = f(-2) + f(-1) + f(0) + f(1) = 2.2$$

$$M_4 = \sum_{i=1}^{n} f\left(\frac{x_{i-1}+x_i}{2}\right) \cdot \Delta x = f(-1.5) + f(-0.5) + f(0.5) + f(1.5) = 2.215$$

Here are the pictures:



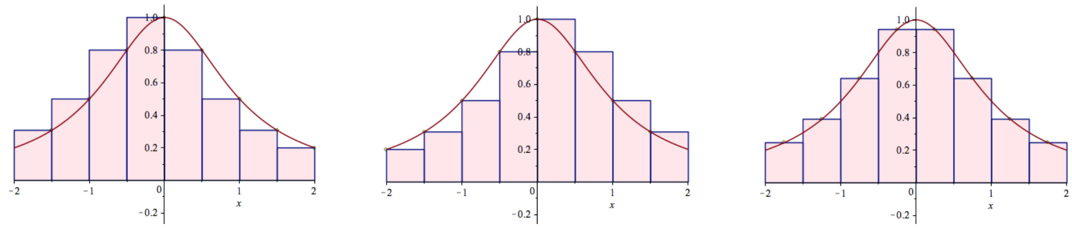
To approximate the area with eight rectangles we let $n = 8$ so that $\Delta x = (2-(-2))/8 = 1/2$ and $x_i = -2 + i \cdot \Delta x = -2 + i/2$. The approximations using right hand endpoints, left hand endpoints, and midpoints are

$$\begin{aligned} R_8 &= \sum_{i=1}^{n} f(x_i) \cdot \Delta x \\ &= f(-3/2)\frac{1}{2} + f(-1)\frac{1}{2} + f(-1/2)\frac{1}{2} + f(0)\frac{1}{2} + f(1/2)\frac{1}{2} + f(1)\frac{1}{2} + f(3/2)\frac{1}{2} + f(2)\frac{1}{2} \\ &= 2.208 \end{aligned}$$

$$L_8 = \sum_{i=1}^{n} f(x_{i-1}) \cdot \Delta x$$

$$= f(-2)\frac{1}{2} + f(-3/2)\frac{1}{2} + f(-1)\frac{1}{2} + f(-1/2)\frac{1}{2} + f(0)\frac{1}{2} + f(1/2)\frac{1}{2} + f(1)\frac{1}{2} + f(3/2)\frac{1}{2}$$
$$= 2.208$$
$$M_8 = \sum_{i=1}^{n} f\left(\frac{x_{i-1} + x_i}{2}\right) \cdot \Delta x$$
$$= f(-7/4)\frac{1}{2} + f(-5/4)\frac{1}{2} + f(-3/4)\frac{1}{2} + f(-1/4)\frac{1}{2} + f(1/4)\frac{1}{2} + f(3/4)\frac{1}{2} + f(5/4)\frac{1}{2} + f(7/4)\frac{1}{2}$$
$$= 2.218$$

And here are the pictures:



We weren't asked for it, but to compute the **exact area** under the graph we let n be arbitrary so that $\Delta x = (2-(-2))/n = 4/n$ and $x_i = -2 + i \cdot \Delta x = -2 + 4i/n$. Then the area under the graph is defined as

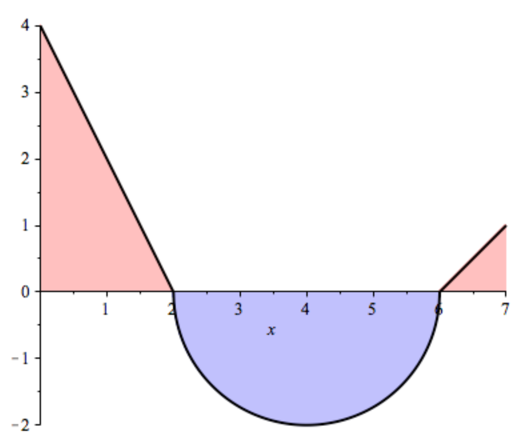
$$\int_{-2}^{2} \frac{1}{1+x^2}\, dx = \lim_{n\to\infty} \left[\sum_{i=1}^{n} \frac{1}{1+(4i/n)^2} \cdot \frac{4}{n} \right] = \lim_{n\to\infty} \left[\sum_{i=1}^{n} \frac{4n}{n^2 + 16i^2} \right].$$

We have no idea how to compute this limit so it doesn't help. However, we will see next week that the antiderivative of $1/(1+x^2)$ is $\arctan(x)$, and then we can use the Fundamental Theorem of Calculus to compute

$$\int_{-2}^{2} \frac{1}{1+x^2}\, dx = \arctan(2) - \arctan(-2) = 2.214.$$

Stay tuned.

4.2.30. The black line in the picture below is the graph of $g(x)$. Compute the integrals $\int_0^2 g(x)\, dx$, $\int_2^6 g(x)\, dx$, and $\int_0^7 g(x)\, dx$.



- $\int_0^2 g(x)\,dx$ is the area of the pink triangle on the left, so

$$\int_0^2 g(x)\,dx = \frac{2\cdot 4}{2} = 4.$$

- $\int_2^6 g(x)\,dx$ is the **negative** of the area of the blue semicircle, so

$$\int_2^6 g(x)\,dx = -\frac{\pi\cdot 2^2}{2} = -6.28.$$

- $\int_0^7 g(x)\,dx$ is the **sum** of the areas of the two pink triangles, **minus** the area of the blue semicircle, so

$$\int_0^7 g(x)\,dx = \frac{2\cdot 4}{2} + \frac{1\cdot 1}{2} - \frac{\pi\cdot 2^2}{2} = 4 + \frac{1}{2} - 6.28 = -1.78.$$

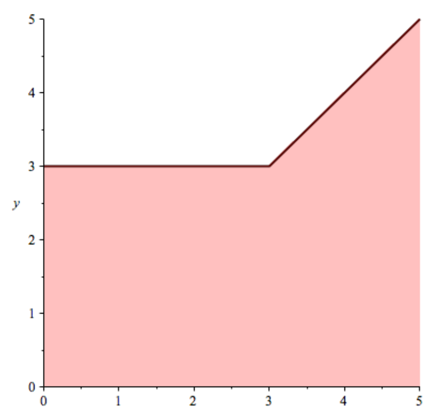
4.2.38. Given that $\int_0^1 3x\sqrt{x^2+4}\,dx = 5\sqrt{5}-8$, what is $\int_1^0 3u\sqrt{u^2+4}\,du$?

This is pretty much a trick question. Your eyes may get confused by all the symbols, but there's really nothing to it. First we switch the limits of integration (which multiplies the result by -1) and then we rename the "dummy variable" from u to x (which doesn't do anything) to get

$$\begin{aligned}\int_1^0 3u\sqrt{u^2+4}\,du &= -\int_0^1 3u\sqrt{u^2+4}\,du \\ &= -\int_0^1 3x\sqrt{x^2+4}\,dx \\ &= -(5\sqrt{5}-8).\end{aligned}$$

4.2.42. Find $\int_0^5 f(x)\,dx$ if $f(x) = \begin{cases} 3 & \text{for } x < 3 \\ x & \text{for } x \geq 3 \end{cases}$.

There are two ways to do this problem. The first way is to draw the graph. Here it is:



Note that the area below the graph from $x = 0$ to $x = 5$ breaks into a rectangle of width 5 and height 3, and a triangle of width 2 and height 2. Therefore,

$$\int_0^5 f(x)\,dx = 5\cdot 3 + \frac{2\cdot 2}{2} = 15 + 2 = 17.$$

The other way to do it is to use the Fundamental Theorem of Calculus. To do this we first break up the interval at $x = 3$. From $x = 0$ to $x = 3$ we have $f(x) = 3$ and from $x = 3$ to $x = 5$ we have $f(x) = x$. Hence

$$\begin{aligned}
\int_0^5 f(x)\,dx &= \int_0^3 f(x)\,dx + \int_3^5 f(x)\,dx \\
&= \int_0^3 3\,dx + \int_3^5 x\,dx \\
&= [3x]_{x=0}^{x=3} + \left[\frac{x^2}{2}\right]_{x=3}^{x=5} \\
&= [3(3) - 3(0)] + \left[\frac{5^2}{2} - \frac{3^2}{2}\right] \\
&= 9 + 8 \\
&= 17.
\end{aligned}$$

This calculation divided up the pink region into a 3 by 3 square (with area 9) from $x = 0$ to $x = 3$ and a trapezoid (with area 8) from $x = 3$ to $x = 5$.

Of course, both methods give the same anwer. Which method do you prefer?

4.3.2. Evaluate $\displaystyle\int_1^2 (4x^3 - 3x^2 + 2x)\,dx$.

Let $f(x) = 4x^3 - 3x^2 + 2x$. One particular antiderivative of this is

$$F(x) = 4\frac{1}{4}x^4 - 3\frac{1}{3}x^3 + 2\frac{1}{2}x^2 = x^4 - x^3 + x^2.$$

Then the F.T.C. gives

$$\begin{aligned}
\int_1^2 (4x^3 - 3x^2 + 2x)\,dx &= \int_1^2 f(x)\,dx \\
&= F(2) - F(1) \\
&= (2^4 - 2^3 + 2^2) - (1^4 - 1^3 + 1^2) \\
&= (16 - 8 + 4) - (1 - 1 + 1) \\
&= 12 - 1 \\
&= 11.
\end{aligned}$$

4.3.6. Evaluate $\displaystyle\int_{-1}^1 t(1-t)^2\,dt$.

Let $f(t) = t(1-t)^2$ and expand to get $f(t) = t(1 - 2t + t^2) = t - 2t^2 + t^3$. One particular antiderivative of this is

$$F(t) = \frac{1}{2}t^2 - 2\frac{1}{3}t^3 + \frac{1}{4}t^4.$$

Then the F.T.C. gives

$$\begin{aligned}
\int_{-1}^1 t(1-t)^2\,dt &= \int_{-1}^1 f(t)\,dt \\
&= F(1) - F(-1)
\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{1}{2}1^2 - \frac{2}{3}1^3 + \frac{1}{4}1^4\right) - \left(\frac{1}{2}(-1)^2 - \frac{2}{3}(-1)^3 + \frac{1}{4}(-1)^4\right) \\
&= \left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) - \left(\frac{1}{2} + \frac{2}{3} + \frac{1}{4}\right) \\
&= \frac{1}{12} - \frac{17}{12} \\
&= -\frac{16}{12} \\
&= -\frac{4}{3}.
\end{aligned}
$$

4.3.10. Evaluate $\displaystyle\int_1^2 \left(x + \frac{1}{x}\right)^2 \, dx$.

Let $f(x) = \left(x + \frac{1}{x}\right)^2$ and expand to get $f(x) = x^2 + 2 + x^{-2}$. One particular antiderivative of this is

$$F(x) = \frac{1}{3}x^3 + 2x + \frac{1}{-1}x^{-1}.$$

Then the F.T.C. gives

$$
\begin{aligned}
\int_1^2 \left(x + \frac{1}{x}\right)^2 \, dx &= \int_1^2 f(x)\, dx \\
&= F(2) - F(1) \\
&= \left(\frac{2^3}{3} + 2(2) - (2)^{-1}\right) - \left(\frac{1}{3} + 2 - 1\right) \\
&= \frac{37}{6} - \frac{4}{3} \\
&= \frac{29}{6}.
\end{aligned}
$$

4.3.14. Evaluate $\displaystyle\int_1^9 \frac{3x - 2}{\sqrt{x}} \, dx$.

Let $f(x) = \frac{3x-2}{\sqrt{x}}$. We can rewrite this as $f(x) = \frac{3x}{\sqrt{x}} - \frac{2}{\sqrt{x}} = 3x^{1/2} - 2x^{-1/2}$. One particular antiderivative of this is

$$F(x) = 3\frac{1}{3/2}x^{3/2} - 2\frac{1}{1/2}x^{1/2} = 3\frac{2}{3}x^{3/2} - 2\frac{2}{1}x^{1/2} = 2x^{3/2} - 4x^{1/2}.$$

Then the F.T.C. gives

$$
\begin{aligned}
\int_1^9 \frac{3x - 2}{\sqrt{x}} \, dx &= \int_1^9 f(x)\, dx \\
&= F(9) - F(1) \\
&= \left(2(9)^{3/2} - 4(9)^{1/2}\right) - \left(2(1)^{3/2} - 4(1)^{1/2}\right) \\
&= (2 \cdot 27 - 4 \cdot 3) - (2 - 4) \\
&= 42 - (-2) \\
&= 44.
\end{aligned}
$$

4.4.6. Use Part 1 of the R.T.C. to find the derivative of $g(x) = \int_1^x (2+t^4)^5 \, dt$.

This is one of those trick questions that looks way harder than it is. If we let $f(x) = (2+x^4)^5$ then $g(x) = \int_1^x f(t) \, dt$ and Part 1 of the F.T.C. says

$$g'(x) = \frac{d}{dx} \int_1^x f(t) \, dt = f(x) = (2+x^4)^5.$$

There's nothing else to say.

4.4.10. Use Part 1 of the F.T.C. to find the derivative of $h(x) = \int_0^{x^2} \sqrt{1+r^3} \, dr$.

This one is slightly tricker, but it's still way easier than it looks. Let $f(x) = \sqrt{1+x^3}$ so that $h(x) = \int_0^{x^2} f(r) \, dr$. Now before we apply Part 1 of the F.T.C. we have to do something about the x^2. We can take care of it by making the substitution $u = x^2$ to get $h(x) = \int_0^u f(r) \, dr$. Then Part 1 of the F.T.C. says

$$\frac{dh}{du} = \frac{d}{du} \int_0^u f(r) \, dr = f(u) = \sqrt{1+u^3} = \sqrt{1+x^6}.$$

But that's not exactly what was asked for. We want $h'(x) = dh/dx$. For this we use the Chain Rule to get

$$\frac{dh}{dx} = \frac{dh}{du} \cdot \frac{du}{dx} = \sqrt{1+x^6} \cdot (2x).$$

[Remark: Problems like **4.4.6** and **4.4.10** are deliberately trying to confuse you. This is very valuable for the learning process, so I think they're good homework problems. However, I will never ask a problem like this on an exam because exams are not for learning.]