Math 161 **Summer 2015**
Homework 1 Solutions **Drew Armstrong**

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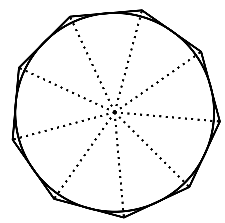
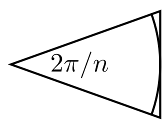
1. Let P_n be a regular polygon with n sides and let C be the largest circle contained inside P_n. Suppose that C has radius r.



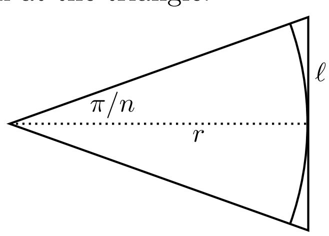
(a) Compute an exact formula for the **perimeter** of P_n.
(b) Compute an exact formula for the **area** of P_n.

[Hint: Divide the polygon into n triangles at its center and consider one of the triangles.



Use the fact that the angle at the center is $2\pi/n$ radians.]

Solution: Let's take a closer look at the triangle:



The radius of the circle divides the triangle into two equal halves. Each half is a right triangle with angle π/n, adjacent side of length r, and opposite side of length ℓ. We conclude that $\tan(\pi/n) = \ell/r$ and hence $\ell = r\tan(\pi/n)$. We don't know the length of the hypotenuse, nor do we care.

For part (a), note that the perimeter of the polygon P_n equals

$$n(2\ell) = n(2r\tan(\pi/n)) = 2rn\tan(\pi/n).$$

For part (b), note that the height of the triangle is r and the base is $2\ell = 2r\tan(\pi/n)$, so the area of the triangle is

$$\frac{1}{2}\cdot \text{height} \cdot \text{base} = \frac{1}{2}\cdot r \cdot 2r\tan(\pi/n) = r^2\tan(\pi/n).$$

Hence the total area of the polygon P_n is

$$n\cdot(\text{area of triangle}) = n(r^2\tan(\pi/n)) = r^2 n\tan(\pi/n).$$

2. (a) Use a calculator to compute the value of $n\,\tan(\pi/n)$ for $n = 1, 10, 100, 1000, 10000$. Now guess the exact value of the limit

$$\lim_{n\to\infty} n\,\tan(\pi/n).$$

(b) Explain how your guess in part (a) agrees with your solution to Problem 1. [Hint: The limit of the perimeter of P_n as n approaches ∞ **should** be the circumference of the circle, i.e., $2\pi r$.]

Solution: For part (a) we define $f(n) = n\,\tan(\pi/n)$ and compute a table of values:

n	1	10	100	1000	10000
$f(n)$	0	3.249196963	3.142626605	3.141602989	3.141592757

For reference, the value of π to 10 places is $\pi = 3.141592654$. Based on this data it is reasonable to guess that

$$\lim_{n\to\infty} n\,\tan(\pi/n) = \pi.$$

For part (b), recall that the perimeter of the polygon P_n is $2rn\,\tan(\pi/n)$. As $n \to \infty$ we expect that this perimeter approaches the circumference of the circle C, hence

$$\lim_{n\to\infty} (\text{perimeter of } P_n) = (\text{circumference of } C) = 2\pi r$$

On the other hand, the formula implies that

$$\begin{aligned}\lim_{n\to\infty} (\text{perimeter of } P_n) &= \lim_{n\to\infty} 2rn\,\tan(\pi/n) \\ &= 2r \cdot \lim_{n\to\infty} n\,\tan(\pi/n)\end{aligned}$$

Putting the two equations together implies that

$$\begin{aligned}2r \cdot \lim_{n\to\infty} n\,\tan(\pi/n) &= 2\pi r \\ \lim_{n\to\infty} n\,\tan(\pi/n) &= \pi,\end{aligned}$$

which agrees with part (a).

3. We showed in class that the region between the graph of $f(x) = x^2$ and the x-axis, from $x = 0$ to $x = 1$, is exactly $1/3$. In this problem you will show that the area between the graph of $g(x) = x^3$ and the x-axis, from $x = 0$ to $x = 1$, is exactly $1/4$.

(a) Draw a picture of this region.

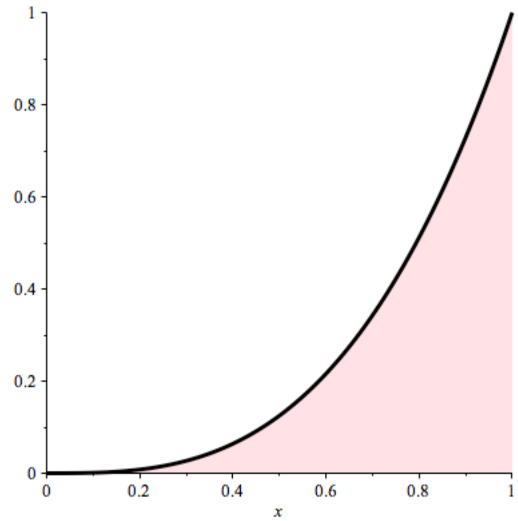
(b) Divide the interval between $x = 0$ and $x = 1$ into n equal intervals of width $1/n$. On the interval from $x = (i-1)/n$ to $x = i/n$ draw a rectangle of height $(i/n)^3$. Write out an expression for the total area of these n rectangles.

(c) Compute the limit of your expression from part (b) as n approaches ∞. [Hint: You should use the algebraic formula

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2.$$

You do not need to say why this mysterious formula is true.]

Solution: For part (a), here is a picture of the region between the graph of $g(x) = x^3$ and the x-axis, from $x = 0$ to $x = 1$:



The fancy name for the area of this region is $\int_0^1 x^3\,dx$. In this problem we will show that $\int_0^1 x^3\,dx = 1/4$.

For part (b), we approximate the region by n recangles, each of width $1/n$. The height of the i-th rectangle is $(i/n)^3$. Hence the total area of the n rectangles is

$$\begin{aligned}
&\frac{1}{n}\left(\frac{1}{n}\right)^3+\frac{1}{n}\left(\frac{2}{n}\right)^3+\frac{1}{n}\left(\frac{3}{n}\right)^3+\cdots+\frac{1}{n}\left(\frac{n-1}{n}\right)^3+\frac{1}{n}\left(\frac{n}{n}\right)^3\\
&=\frac{1}{n}\left[\left(\frac{1}{n}\right)^3+\left(\frac{2}{n}\right)^3+\left(\frac{3}{n}\right)^3+\cdots+\left(\frac{n-1}{n}\right)^3+\left(\frac{n}{n}\right)^3\right]\\
&=\frac{1}{n}\left[\frac{1^3}{n^3}+\frac{2^3}{n^3}+\frac{3^3}{n^3}+\cdots+\frac{(n-1)^3}{n^3}+\frac{n^3}{n^3}\right]\\
&=\frac{1}{n}\cdot\frac{1}{n^3}\left[1^3+2^3+3^3+\cdots+(n-1)^3+n^3\right]\\
&=\frac{1}{n^4}\left[1^3+2^3+3^3+\cdots+(n-1)^3+n^3\right].
\end{aligned}$$

For part (c), we expect that as the number of rectangles approaches ∞, the area of the triangles approaches the area of our region. Then using the mysterious formula gives

$$\begin{aligned}
\int_0^1 x^3\,dx &= \lim_{n\to\infty}\frac{1}{n^4}\left[1^3+2^3+3^3+\cdots+(n-1)^3+n^3\right]\\
&=\lim_{n\to\infty}\frac{1}{n^4}\left[\frac{1}{4}n^4+\frac{1}{2}n^3+\frac{1}{4}n^2\right]\\
&=\lim_{n\to\infty}\left[\frac{1}{4}+\frac{1}{2n}+\frac{1}{4n^2}\right]\\
&=\lim_{n\to\infty}\frac{1}{4}+\lim_{n\to\infty}\frac{1}{2n}+\lim_{n\to\infty}\frac{1}{4n^2}\\
&=\frac{1}{4}+0+0\\
&=\frac{1}{4}.
\end{aligned}$$