Smale's Mean Value Conjecture

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The lecture is a survey of classic conjectures in the geometry of polynomials, starting with the famous Mean Value Conjecture of S. Smale, which states:

Conjecture 1 (Smale) *Let $p(z)$ be a polynomial of degree n such that $p(0) = 0$ and $p'(0) \neq 0$. Then*

$$\min \left\{ \left| \frac{p(\zeta)}{\zeta p'(0)} \right| : \ p'(\zeta) = 0 \right\} \leq K, \tag{1}$$

where $K = 1$ or probably $(n-1)/n$.

In 1958, the author formulated the following:

Conjecture 2 *Let $p(z) = (z - \zeta_1)(z - \zeta_2) \cdots (z - \zeta_n)$ be a polynomial of degree $n \geq 2$ with all its zeros on the closed unit disk. Then each closed disk with center ζ_k; $k = 1, 2, \ldots, n$ and radius 1 contains a critical point of $p(z)$*

Until now, the two conjectures are not proved in general, see [1, p. 214 - 240]. Hundreds of papers dedicated to the conjectures employ usually the classical instruments as the Gauss - Lucas theorem and a polarity, where the position of the critical points of a polynomial is located from the fixed position of its zeros. The motivation of the lecture is to demonstrate a reverse approach. The critical points of the polynomial are fixed and the zeros are functions of the constant term of the polynomial.

References

[1] Rahman, Q. I. and Schmeisser, G., *Analytic Theory of Polynomials*, Oxford Univ. Press Inc., New York, (2002).