Spacetime Intrinsic Flat Convergence

Christina Sormani (CUNYGC and Lehman College)

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joint work with Carlos Vega (SUNY Binghamton)
and Anna Sakovich (Uppsala University Sweden)

A Celebration of Mathematical Relativity
in Miami, Dec 14-16,2018

Thank You for the Opportunity to Speak

I've known Greg for many years, since I was a doctoral student working on Riemannian Geometry in the 1990's.

I am particularly grateful to Greg and to Jim, Gerhardt, Rick and Piotr for inviting me to serve as a visiting research professor at MSRI in 2013. That was a career altering opportunity for me.

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Happy Birthday Greg!

Thank you for all the Advice and Mentoring!

What is Spacetime?

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Light travels along null geodesics γ s.t. $g(\gamma'(s), \gamma'(s)) = 0 \quad \forall t$.

Every event has a light cone of null vectors oriented with a future and a past.

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This is a flat spacetime with no gravity.

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In General Relativity: Spacetime is a manifold M^4 endowed with a Lorentzian metric g of signature $- + + +$.

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Example: $M^4 = (0, \pi) \times \mathbb{S}^3$ with $g = -dt^2 + \sin^2(t) g_{\mathbb{S}^3}$.
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Warning: A time function need not exist! Example: M=$\mathbb{T}^4$

Friedmann–Lemaître–Robertson–Walker Spacetimes

FLRW spacetimes are used by cosmologists to model the universe.

One assumes that

$$M^4 = (a, b) \times N^3 \text{ and } g = -dt^2 + f^2(t) g_N$$

where N is a homogeneous Riemannian manifold of constant curvature, K.

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While these models are very important in cosmology... they are an oversimplification of the observed universe.
One may ask: How close is the true universe to these models?

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FLRW spacetimes are over simplified:

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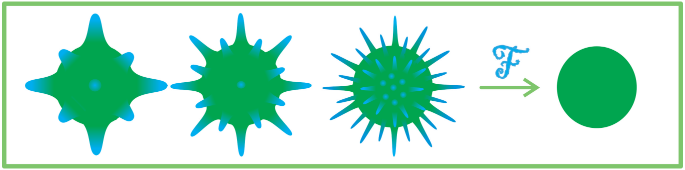
because N^3 has constant curvature

and matter is assumed to be distributed evenly.

Yet the real universe is known to have stars with gravity wells:



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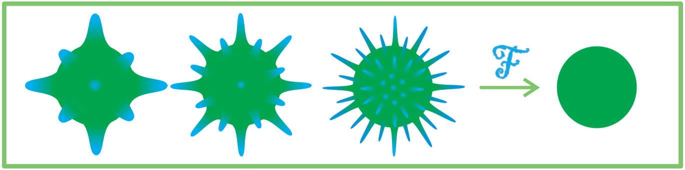

Is a large round universe filled with stars approximately a sphere?

And what about black holes?
Even a universe with a single black hole cannot be considered to be close to an FLRW space, unless perhaps one cuts out the interior of the black hole along the horizon.

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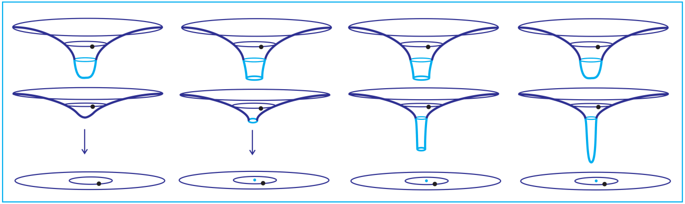

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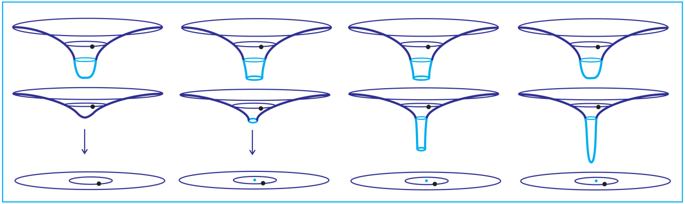

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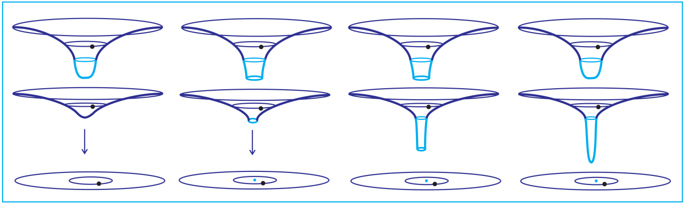

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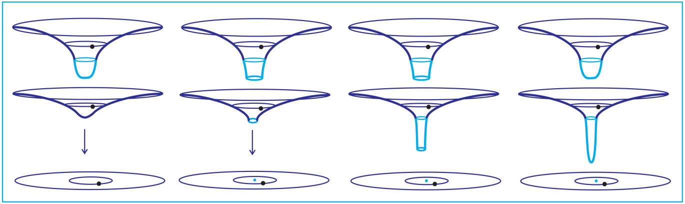

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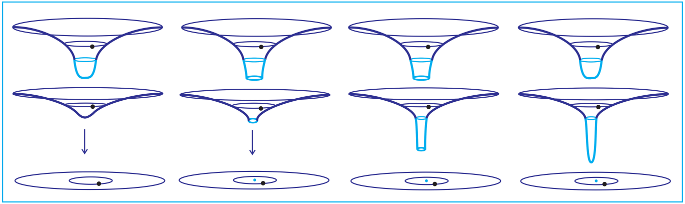

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The Spacetime Intrinsic Flat Distance: $d_{\mathcal{SF}}(M_1, M_2)$

Joint with Wenger [JDG2011]: The intrinsic flat distance $d_{\mathcal{F}}(M_1, M_2)$ is defined between a pair of rectifiable metric spaces. It is defined by taking the infimum over all distance preserving maps, $\varphi_j : M_j \to Z$ into all complete metric spaces, Z, of the Federer-Flemming flat distance between the images $\varphi_j(M_j) \subset Z$.

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Discussion with Andersson and Yau: Try to convert spacetimes, (M, g), canonically into metric spaces, (M, d), then take

$$d_{\mathcal{SF}}((M_1, g_1), (M_2, g_2)) = d_{\mathcal{F}}((M_1, d_1), (M_2, d_2)).$$

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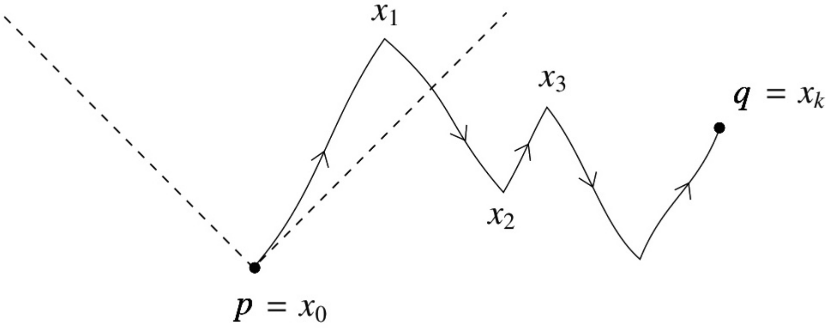
Joint with Vega [CQG2016]: We introduce the null distance, $\hat{d}_\tau$, on a spacetime, (M, g), endowed with a time function, τ.
(a time function is strictly increasing along future causal curves)

The Null Distance between events in a Spacetime: $\hat{d}_\tau(p,q)$

Joint with Vega: Given a time function, τ, on a spacetime, (M,g),

$$\hat{d}_\tau(p,q) = \inf_\beta \hat{L}_\tau(\beta) = \inf_\beta \sum_{i=1}^{k} |\tau(\beta(t_i)) - \tau(\beta(t_{i+1})|$$

where the inf is over all piecewise causal curves β from p to q, which are causal from $x_i = \beta(t_i)$ to $x_{i+1} = \beta(t_{i+1})$:



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Example: Minkowski Spacetime

The metric tensor is
$g = -dt^2 + dx_1^2 + dx_2^2$
So if we take $\tau = t$
then the level sets of
$\hat{d}_\tau(p, \cdot)$ are cylinders
aligned perfectly with
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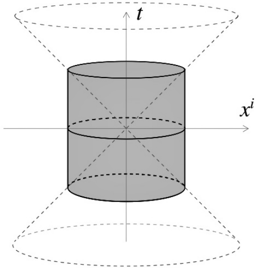

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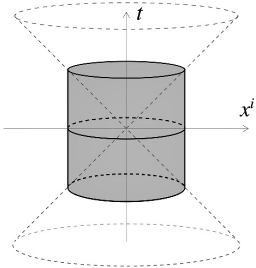

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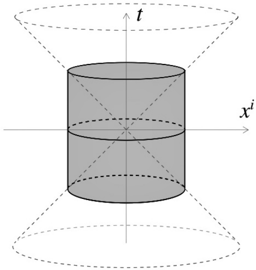

If we take $\tau = t^3$ then $\hat{d}_\tau$ is not even a definite metric!

When the Null Distance Encodes Causality it is Definite

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Lemma: p is in the future of $q \implies \hat{d}_\tau(p,q) = \tau(p) - \tau(q)$.

Definition: We say $\hat{d}_\tau$ encodes causality when this is $\iff$.

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Thm: If $\hat{d}_\tau$ encodes causality then $(M, \hat{d}_\tau)$ is a metric space.

Lemma: $\hat{d}_\tau$ encodes causality on an FLRW spacetime $\tau = t$.

So we can measure the $d_{\mathcal{SIF}}$ between two FLRW spacetimes by converting them to metric spaces and taking the $d_{\mathcal{F}}$ between them.
But what about other spacetimes?

The Cosmological Time Function

Andersson-Galloway-Howard defined a time function which is independent of a particular gauge on a given spacetime (see also Wald-Yip):

Defn: $\tau_{AGH}(p)$ is the supremum of the Lorentz distance from p over all points q in its past. That is,

$$\tau_{AGH}(p) = \sup_{q \leq p} \int_c \sqrt{-g(c'(s), c'(s))}\, ds$$

where c is a future causal curve from q to p. It is said to be "regular" if it is finite on all of M and converges to 0 on all past inextensible curves.

With Vega: If one defines the null distance using a regular cosmological time function, $\tau = \tau_{AGH}$, then it is a definite metric: $\hat{d}_\tau(p,q) = 0 \iff p = q$.

Open: Does it also encode causality? Are the charts biLip?
Work in progress in this direction by B Allen and A Burtscher.

Spacetime Intrinsic Flat Distances

between Big Bang Spacetimes [in progress with Vega]

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The classic Friedmann-Lemaître-Robertson-Walker spacetimes are warped product manifolds, with metric tensors $g = -dt^2 + f^2(t)h$.

They have a big bang iff $t > 0$ and $\lim_{t \to 0^+} f(t) = 0$.

Thm: $\exists$ a single big bang point, p_0, s.t. $\hat{d}_\tau(q, p_0) = \tau(q) \, \forall q \in M$.

We can then generalize the definition of big bang spacetimes to include all spacetimes with such a big bang point.

We then convert all such (M, g) into pointed metric spaces $(M, \hat{d}_\tau, p_0)$ canonically and uniquely.

We can then describe their spacetime intrinsic flat distance and the pointed intrinsic flat convergence of such spaces.

Thus we can achieve our goal: to understand what it means for the universe to be close to an FLRW space.

Spacetime Intrinsic Flat Distances between

Maximal Developments [in progress with Sakovich]

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Maximal Developments [in progress with Sakovich]

We consider spacetimes which are maximal developments of initial data sets solving Einstein's Equations [Choquet-Bruhat&Geroch].

Example: The Schwarzschild spacetime of mass $m > 0$ is

$$g_{Sch,m} = -\left(\frac{r^2 - 2mr}{r^2}\right) dt^2 + \left(\frac{r^2}{r^2 - 2mr}\right) dr^2 + r^2 g_{\mathbb{S}^2} \text{ with } r > 2m.$$

Here we have cut out the interior of the black hole along the horizon at $r = 2m$. The region $t > 0$ is the maximal development of the $t = 0$ level.

We study the cosmological time, $\tau = \tau_{AGH}$, and null distance, $\hat{d}_\tau$ on the $t > 0$ regions of Schwarzschild spacetimes (and Kerr spacetimes). We prove the spacetime intrinsic flat limits of these regions as $m \to 0$ is the $t > 0$ region in Minkowski spacetime.

Next we plan to study far more general maximal developments.

Call for more study of the cosmological time!

What spaces have a regular cosmological times?

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These are questions for Lorentzian Geometers!!!

Meanwhile there is work on $\mathcal{F}$

Joint with Wenger [JDG2011]: The intrinsic flat distance $d_{\mathcal{F}}(M_1, M_2)$ is defined between a pair of Riemannian manifolds. It is defined by taking the infimum over all distance preserving maps, $\varphi_j : M_j \to Z$ into all complete metric spaces, Z, of the Federer-Flemming **flat** distance between the images $\varphi_j(M_j) \subset Z$.

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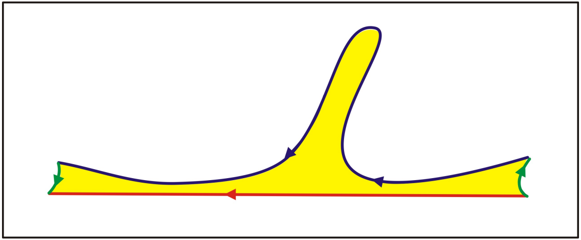
The **flat** distance between two submanifolds, $T_1, T_2 \subset Z$ is

$$d_F^Z(T_1, T_2) \leq \inf\{\mathrm{vol}_m(\boxed{A}) + \mathrm{vol}_{m+1}(\boxed{B}) : \boxed{A} + \partial \boxed{B} = T_1 - T_2\},$$



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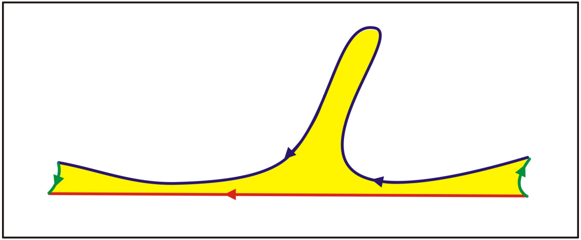

Results on $\mathcal{F}$ convergence:

Joint with Lakzian: Methods of estimating $d_{\mathcal{F}}(M_1, M_2)$ by finding $U_i \in M_i$ which are biLip close, and showing $vol(M_i \setminus U_i)$ is small, and $area(\partial U_i)$ and distance differences λ are controlled.

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Wenger Compactness: If M_i have

$$vol(M_i) \leq V \quad diam(M_i) \leq D \quad area(\partial M_i) \leq A$$

then a subsequence $M_{i_j} \xrightarrow{\mathcal{F}} M_\infty$ possibly 0.

This was applied with Huang and Lee combined with:

Sormani Arzela-Ascoli: If $F_i : M_i \to W$ where W is compact have $Lip(F_i) \leq K$ and $M_i \xrightarrow{\mathcal{F}} M_\infty$ then there exists $F_\infty : M_\infty \to W$ with $Lip(F_\infty) \leq K$.

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Recent joint work with Brian Allen: provides controls on the distances and metric tensors which imply GH and $\mathcal{F}$ convergence.

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Thanks again! Happy Birthday, Greg!!!