Quantum strong energy inequality and the Hawking singularity theorem

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Introduction

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Singularity theorems structure (Senovilla 1998)

1. **Causality condition**
   e.g. There is a Cauchy surface $\mathscr{H}$: complete spacelike C^∞ hypersurface that intersects every null and timelike line only once
2. **The initial or boundary condition**
   e.g. There exists a trapped surface: spacelike hypersurface for which two null normals have negative expansion
3. **The energy condition**
   e.g. Null Energy Condition (Penrose)
   $R_{ab}\ell^a\ell^b \geq 0$ with ℓ^a: null
   Strong Energy Condition (Hawking)
   $R_{ab}U^aU^b \geq 0$ with U^a :timelike

$\Rightarrow$ Then the spacetime is geodesically incomplete.

Raychaudhuri equation

- Expansion

$$\dot{\theta} = -\frac{1}{n-1}\theta^2 - 2\sigma^2 - R_{ab}U^aU^b$$

- Shear scalar
- Curvature

Proof structure:

- Initial condition: Geodesics start focusing
- Energy condition: Focusing continues
- Causality condition: No focal points

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Energy conditions and quantum inequalities

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Average energy conditions bound the weighted energy density along an entire geodesic

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Quantum Inequalities introduce a restriction on the possible magnitude and duration of any negative energy densities or fluxes within a quantum field theory.

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$$\int d\tau f^2(\tau)\langle : \rho : \rangle_\omega(\gamma(\tau)) \geq -A$$

A singularity theorem with a weakened energy condition

(Fewster, Galloway 2011)

1. **Energy condition**

$$\int_{-\infty}^{\infty} r(\tau) f(\tau)^2 d\tau \geq -|||f|||^2$$

- $r(\tau) = R_{\mu\nu} U^\mu U^\nu$
- $|||f|||^2 = \sum_{\ell=0}^{L} Q_\ell ||f^{(\ell)}||^2$

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$\Rightarrow$ If the geodesic is complete, the Raychaudhuri equation has no solution ($\theta \rightarrow -\infty$). So the geodesic is incomplete.

The non-minimally coupled field

The nonminimally-coupled scalar field obeys the field equation

$$P_\xi \phi = 0, \qquad P_\xi := \Box_g + m^2 + \xi R$$

where ξ is the coupling constant.

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$$T_{\mu\nu} = (\nabla_\mu \phi)(\nabla_\nu \phi) + \frac{1}{2} g_{\mu\nu}(m^2\phi^2 - (\nabla\phi)^2) + \xi(g_{\mu\nu}\Box_g - \nabla_\mu \nabla_\nu - G_{\mu\nu})\phi^2$$

Effective energy density (EED) on a timelike geodesic γ

$$\rho = T_{\mu\nu}\dot{\gamma}^\mu \dot{\gamma}^\nu - \frac{1}{n-2} T \, .$$

Average strong energy condition

$$\int_\gamma d\tau \rho f^2(\tau) = \int_\gamma d\tau \Bigg\{ -\frac{1-2\xi}{n-2} m^2 f^2(\tau) + \left(1 - 2\xi \frac{n-1}{n-2}\right) (\nabla_{\dot\gamma}\phi)^2 f^2(\tau)$$
$$+\frac{2\xi}{n-2} h^{\mu\nu}(\nabla_\mu\phi)(\nabla_\nu\phi) f^2(\tau) + 2\xi[\nabla_{\dot\gamma}(f(\tau))\phi]^2 - 2\xi\phi^2(f'(\tau))^2$$
$$-\xi R_{\mu\nu}\dot\gamma^\mu\dot\gamma^\nu f^2(\tau) + \frac{2\xi^2}{n-2} R\phi^2 f^2(\tau) \Bigg\}$$

$$\xi \in [0, \xi_c]$$

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$$\overset{+}{\boxed{+\frac{2\xi}{n-2} h^{\mu\nu}(\nabla_\mu\phi)(\nabla_\nu\phi) f^2(\tau)}} \overset{+}{\boxed{+2\xi[\nabla_{\dot\gamma}(f(\tau))\phi]^2}} \overset{-}{\boxed{-2\xi\phi^2(f'(\tau))^2}}$$

$$\boxed{-\xi R_{\mu\nu}\dot\gamma^\mu\dot\gamma^\nu f^2(\tau)} \boxed{+\frac{2\xi^2}{n-2} R\phi^2 f^2(\tau)} \Bigg\}$$

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$$\int_\gamma d\tau\, \rho\, f^2(\tau) \geq -\int_\gamma d\tau \Big\{ \frac{1-2\xi}{n-2} m^2 f^2(\tau) + \xi\Big(2(f'(\tau))^2 + R_{\mu\nu}\dot\gamma^\mu\dot\gamma^\nu f^2(\tau) - \frac{2\xi^2}{n-2} R f^2(\tau)\Big)\Big\}\phi^2$$

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Imposing Einstein's equation

$$8\pi\rho = R_{\mu\nu}\dot{\gamma}^{\mu}\dot{\gamma}^{\nu}\,, \qquad \left(\frac{n}{2} - 1\right) R = 8\pi T\,.$$

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If ϕ obeys global bounds $|\phi| \leq \phi_{\text{max}}$ and $|\nabla_{\dot{\gamma}}\phi| \leq \phi'_{\max}$

$$\int R_{\mu\nu}\dot{\gamma}^\mu\dot{\gamma}^\nu f(\tau)^2\, d\tau \geq -Q(\|f'\|^2 + \tilde{Q}^2\|f\|^2),$$

with Q, $\tilde{Q}$ depend on m, ξ, ϕ_{max} and ϕ'_{max}.

The singularity theorem

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   (i) There is $K > 0$ so that

$$\dot{\theta}|_{\gamma(\tau)} + \frac{\theta(\gamma(\tau))^2}{n-1} \geq -Q(K^2 + \tilde{Q}^2) \qquad \text{on } (-\tau_0, 0]$$

holds along every future-directed unit-speed geodesic $\gamma(\tau)$ issuing orthogonally from S at $\tau = 0$, and
(ii) the expansion θ on S obeys

$$\theta|_S < -\tilde{Q}\sqrt{Q(n-1) + Q^2/2} - \frac{1}{2} QK \coth(K\tau_0) \, .$$

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$\Rightarrow$ Then (M,g) is future timelike geodesically incomplete.

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- Quantized scalar field in Minkowski spacetime of dimension 4, in a thermal state of temperature $T < T_m$, $T_m = mc^2/k$
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Pion: $m = 140\mathrm{MeV}/c^2$, $\theta_0 \sim 10^{-19} s^{-1}$ and temperature up to $T = 10^{10}\mathrm{K}$
Higgs: $m = 125\mathrm{GeV}/c^2$, $\theta_0 \sim 10^{-14}\mathrm{s}^{-1}$ and temperature up to $T = 10^{13}\mathrm{K}$

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$\Rightarrow$ When the field mass is taken equal to an elementary particle we need very little initial contraction for either geodesic incompleteness or that, the solution evolves to a temperature approaching that of the early universe.

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- Introduction of a unital *-algebra $\mathscr{A}(M)$ on our manifold M, generated by the objects $\Phi(f)$

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- We only consider Hadamard states on our algebra, the two-point function $W(x, y) = \langle \Phi(x)\Phi(y) \rangle_\omega : \mathscr{D}(M) \times \mathscr{D}(M) \to \mathbb{C}$ has a prescribed singularity structure so that the difference between two states is smooth.

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- The smeared local Wick polynomials of the form

$$\langle :\nabla^{(r)}\Phi\nabla^{(s)}\Phi:_\omega(f)\rangle_{\omega'} = T^{r,s}[f](W' - W),$$

are part of an extended algebra

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- We need a prescription for finding algebra elements that qualify as local and covariant Wick powers. Hollands and Wald (2014) set out a list of axioms that we follow
- While the quadratic normal ordered expressions obey Leibniz' rule, but not generally the field equation, the differences in their expectation values obey both

$$\langle (\nabla^{(r)}\Phi P_\xi \Phi)(f)\rangle_{\omega'} - \langle (\nabla^{(r)}\Phi P_\xi \Phi)(f)\rangle_{\omega} = 0.$$

- Expectation value of the quantized EED

$$\langle :\rho_U:_\omega(f)\rangle_{\omega'} = \langle \rho_U(f)\rangle_{\omega'} - \langle \rho_U(f)\rangle_{\omega}$$

Quantum strong energy inequality (QSEI)

Theorem

For non-minimally coupled scalar field with coupling constant $\xi \in [0, \xi_c]$, γ a timelike geodesic, for all Hadamard states ω, the normal-ordered effective energy density obeys the SQEI

$$\int d\tau f^2(\tau)\langle :\rho_U: \rangle_\omega(\gamma(\tau)) \geq - \left[\mathfrak{Q}_A(f)\mathbb{1} + \langle :\Phi^2: \circ \gamma\rangle_\omega(\mathfrak{Q}_B(f) + \mathfrak{Q}_C(f))\right] ,$$

where

$$\mathfrak{Q}_A(f) = \int_0^\infty \frac{d\alpha}{\pi} \left(\phi^*(\hat{\rho}_1 W_0)(\bar{f}_\alpha, f_\alpha) + 2\xi\alpha^2\phi^* W_0(\bar{f}_\alpha, f_\alpha)\right) ,$$

$$\mathfrak{Q}_B[f](\tau) = \frac{1-2\xi}{n-2} m^2 f^2(\tau) + 2\xi(f'(\tau))^2 ,$$

and

$$\mathfrak{Q}_C[f](\tau) = f^2(\tau)\xi \left(R_{\mu\nu}U^\mu U^\nu - \frac{2\xi}{n-2}R\right)(\tau) .$$

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- $\boxed{\mathfrak{Q}_A(f)}$: *State independent terms*
- $\boxed{\mathfrak{Q}_B(f)}$: *State dependent terms*
- $\boxed{\mathfrak{Q}_C(f)}$: *State dependent curvature terms*

Singularity theorem hypothesis from QSEI

If we constrain the state ω and the metric $g_{\mu\nu}$ to those that satisfy the semiclassical Einstein equation we can convert the QEI to a curvature condition

$$\langle : T_{\mu\nu} : \rangle_\omega = 8\pi G_{\mu\nu} \,.$$

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Problems

1. The semiclassical Einstein equation requires that the stress-energy tensor is Hadamard renormalized
2. In curved spacetimes there is no preferred state

For minimally coupled fields in Minkowski

$$\int d\tau f^2(\tau) R_{\mu\nu}\dot{\gamma}^\mu\dot{\gamma}^\nu \geq -8\pi\left[\int_0^\infty \frac{d\alpha}{\pi}\phi^*((\nabla_U \otimes \nabla_U)\, W_0)(\bar{f}_\alpha, f_\alpha) + \frac{\mu^2}{n-2}\langle :\Phi^2: \circ\, \gamma\rangle_\omega(f^2)\right] .$$

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- Even number of dimensions
- Restrict to a class of Hadamard states ω for which the field's magnitude has a finite maximum magnitude

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$$\int d\tau f^2(\tau) R_{\mu\nu}\dot{\gamma}^\mu \dot{\gamma}^\nu \geq -\frac{8\pi S_{2m-2}}{2m(2\pi)^{2m}} ||f^{(m)}||^2 - \frac{8\pi\mu^2\phi_*^2}{2m-2} ||f||^2 \,.$$

Singularity theorem hypothesis from QSEI

- The result applies in curved spacetimes only if the support of the sampling function is constrained to be small compared to local curvature length scales.
- To discuss averages over long timescales we will use a partition of unity. We define bump functions ϕ_n each supported only on an interval $2\tau_0$.
- We obtain a sum of integrals, each of which can be bounded using the Minkowski result

$$\int_{-\infty}^{\infty} R_{\mu\nu}\dot{\gamma}^{\mu}\dot{\gamma}^{\nu} f^2(\tau) d\tau \geq -\frac{8\pi S_{2m-2}}{2m(2\pi)^{2m}} \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \boxed{[(f\phi_n)^{(m)}]^2} d\tau - \frac{8\pi\mu^2\phi_*^2}{2m-2} \|f\|^2 .$$

Singularity theorem hypothesis from QSEI

$$\int_{-\infty}^{\infty} R_{\mu\nu}\dot{\gamma}^{\mu}\dot{\gamma}^{\nu} f^2(\tau)d\tau \geq -Q_m(\|f^{(m)}\|^2 + \tilde{Q}_m^2\|f\|^2) := |||f|||^2 ,$$

where Q_m and $\tilde{Q}_m$ constants that depend on: m, μ, ϕ_* and the maximum value of the bump function and its derivatives.

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where Q_m and $\tilde{Q}_m$ constants that depend on: m, μ, ϕ_* and the maximum value of the bump function and its derivatives.
This is an expression of the form

$$\int_{-\infty}^{\infty} r(\tau) f(\tau)^2 d\tau \geq -|||f|||^2$$

so we can prove a singularity theorem with this condition.

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$$\theta|_S < -L(Q_m, \tilde{Q}_m) - M(Q_m, \tilde{Q}_m, K, \tau_0)\,.$$

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Conclusions

- Classical singularity theorems have easily violated energy conditions in their hypotheses
- Derived a Hawking-type singularity theorem with an energy condition obeyed by the classical non-minimally coupled Einstein-Klein-Gordon field
- Developed a strong quantum energy inequality for the non-minimally coupled scalar field
- Proved a singularity theorem with an energy condition derived by a QEI obeyed by the minimally coupled quantum scalar field that obeys the semiclassical Einstein equation
- Work in progress: prove an absolute (Hadamard renormalised) QSEI for spacetimes with curvature
- Future work: Penrose-type theorem