On Bartnik mass estimates of CMC Bartnik data

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GregFest
December 14-16, 2018
Miami, USA

Plan for the talk:

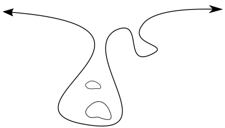
- Initial data sets
- Hawking and Bartnik quasi-local masses
- Bartnik mass computation and estimates
- Mantoulidis–Schoen construction adapted for CMC data

A Riemmanian manifold (M^3, γ) is a (time-symmetric) **initial data set** satisfying the dominant energy condition if

$$R(\gamma) \geq 0,$$

where $R(\gamma)$ is the scalar curvature.

We will assume that (M^3, γ) is **asymptotically flat (AF)**, meaning that $M \setminus K \cong \mathbb{R}^3 / B_1(0)$ and $g \to g_E$ (the Euclidean metric) at infinity sufficiently fast. In addition, $R(\gamma) \in L^1(M)$.



Total mass: The ADM mass, $m_{\text{ADM}}(g)$, is measured at infinity. Two AF initial data sets coinciding outside a compact set, have the same ADM mass.

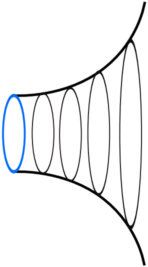
Let $m > 0$. The **Schwarzschild manifold of mass** m is the Riemannian manifold $((2m, \infty) \times \mathbb{S}^2, \gamma_m)$ with

$$\gamma_m = \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 g_*,$$

where g_* is the standard round metric on $\mathbb{S}^2$.

Some important properties:

- Asymptotically flat,
- $R(\gamma_m) = 0$,
- $m_{\mathrm{ADM}} = m$, and
- The boundary $r = 2m$ is an outer-minimizing horizon (minimal surface $H = 0$).



Positive Mass Theorem

Theorem (Schoen and Yau 1979, 2017, Witten 1981). Let (M, γ) be a complete asymptotically flat manifold with $R(\gamma) \geq 0$. Then $m_{\mathrm{ADM}}(\gamma) \geq 0$. Equality holds if and only if (M, γ) is isometric to the Euclidean space.

Riemannian Penrose Inequality

Theorem (Huisken and Ilmanen 2001, Bray 2001). Let (M, γ) be an asymptotically flat manifold with $R(\gamma) \geq 0$ and such that its boundary ∂M consists of minimal outer-minimizing closed surfaces, then

$$m_{\mathrm{ADM}}(g) \geq \frac{1}{2}\sqrt{\frac{|\partial M|_{\gamma|_{\partial M}}}{4\pi}}. \tag{RPI}$$

Equality holds if and only if (M, γ) is isometric to a Schwarzschild manifold.

Let Σ be a closed surface inside an initial data set (M, γ). Let H denote the mean curvature of Σ in M and g denote γ restricted to Σ. Then the **Hawking mass**, $\mathfrak{m}_H(\Sigma, g, H)$, is defined as

$$\mathfrak{m}_H(\Sigma, g, H) = \sqrt{\frac{|\Sigma|_g}{16\pi}} \left(1 - \frac{1}{16\pi} \int_\Sigma H^2 \, d\sigma\right).$$

Remark: Notice that we can rewrite the RPI as

$$m_{\mathrm{ADM}} \geq \mathfrak{m}_H(\partial M, g = \gamma|_{\partial M}, H = 0).$$

Let $(\Sigma \cong \mathbb{S}^2, g, H)$ be given *Bartnik data*, i.e., $H \geq 0$ is a smooth function on Σ. Define the following set:

$$\mathcal{A} = \left\{ (M,\gamma) : \begin{array}{c} (M,\gamma) \text{ AF, } R(\gamma) \geq 0, \ \partial M \text{ is outer-minimizing,} \\ (\partial M, \gamma|_{\partial M}) \cong (\Sigma, g), \text{ and } H_{\partial M} = H \end{array} \right\}$$

The **Bartnik mass**, $\mathfrak{m}_B(\Sigma, g, H)$, is defined as

$$\mathfrak{m}_B(\Sigma, g, H) = \inf\{m_{\mathrm{ADM}}(\gamma) : \gamma \in \mathcal{PM}\}.$$

Remarks:

(1) (RPI) $\Rightarrow \mathfrak{m}_H(\Sigma, g, H \equiv 0) \leq \mathfrak{m}_B(\Sigma, g, H \equiv 0)$.

(2) Any $(M,\gamma) \in \mathcal{A}$ gives an upper bound for the Bartnik mass: $\mathfrak{m}_B(\Sigma, g, H) \leq m_{\mathrm{ADM}}(\gamma)$.

Theorem (Mantoulidis and Schoen, 2016). Given minimal Bartnik data $(\Sigma \cong \mathbb{S}^2, g_o, H_o = 0)$, with $\lambda_1(-\Delta_{g_o} + K(g_o)) > 0$,

$$\mathfrak{m}_B(\Sigma, g_o, H_o = 0) = \mathfrak{m}_H(\Sigma, g, H_o = 0).$$

Theorem ($\star$) (C., Cederbaum, McCormick and Miao, 2017). Given CMC Bartnik data ($H_o > 0$ constant) $(\Sigma \cong \mathbb{S}^2, g_o, H_o)$ with $K(g_o) > 0$, there exist constants $\alpha \geq 0$ and $0 < \beta \leq 1$ (depending on g_o: $\alpha = 0$ and $\beta = 1$ iff g_o is round) such that if

$$\frac{H_o^2 r_o^2}{4} < \frac{\beta}{1+\alpha}, \text{ where } r_o = \sqrt{\frac{|\Sigma|_{g_o}}{4\pi}},$$

then,

$$\mathfrak{m}_B(\Sigma, g_o, H_o) \leq (1 + \mathcal{E}(\alpha, \beta, H_o))\, \mathfrak{m}_H(\Sigma, g_o, H_o).$$

Theorem (Miao, Wang and Xie, 2018).
Let $(\Sigma \cong \mathbb{S}^2, g_o, H_o)$ be CMC Bartnik data with $K(g_o) > 0$, that arises as the boundary of a compact 3-manifold with non-negative scalar curvature. Then there exists a constant $\zeta \geq 0$ (depending on g_o: $\zeta = 0$ iff g_o is round), such that

$$\mathfrak{m}_B(\Sigma \cong \mathbb{S}^2, g_o, H_o) \leq \frac{3}{2} r_o (1 + \tau\zeta)\tau\zeta + \mathfrak{m}_H(\Sigma, g_o, H_o),$$

where $r_o = \sqrt{\dfrac{|\Sigma|_{g_o}}{4\pi}}$ and $\tau = \frac{1}{2} r_o H_o$.

Adapted Mantoulidis–Schoen (for $(\star)$)

(Σ, g_o)
with $K(g_o) > 0$

Adapted Mantoulidis–Schoen (for $(\star)$)

Find a smooth path of metrics $\{g(t)\}_{0 \leq t \leq 1}$:
$g(0) = g_o$, $g(1)$ round, $K(g(t)) > 0$, $\mathrm{tr}_{g(t)} g'(t) = 0$
and $g'(t) = 0$ for $t \in [\theta, 1]$, for $0 < \theta < 1$

(Σ, g_o)
with $K(g_o) > 0$

Adapted Mantoulidis–Schoen (for $(\star)$)

Find a smooth path of metrics $\{g(t)\}_{0\leq t\leq 1}$:
$g(0)=g_o$, $g(1)$ round, $K(g(t))>0$, $\mathrm{tr}_{g(t)}g'(t)=0$ for all t,
and $g'(t)=0$ for $t\in[\theta,1]$, for $0<\theta<1$

We define the following scaling invariant quantities **(Miao and Xie, 2017)**

$$\alpha := \max_{[0,1]\times\Sigma} |g'(t)|^2_{g(t)},$$

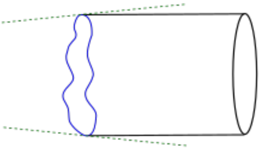
and

$$\beta := \min_{[0,1]\times\Sigma} r_o^2 K(g(t)).$$

These quantities measure "how round (Σ, g_o) is".

Adapted Mantoulidis–Schoen (for $(\star)$)

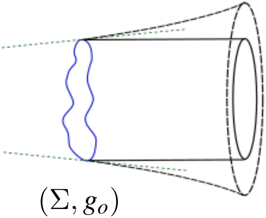
Prescribe H:



(Σ, g_o) $(\Sigma, r_o^2 g_*)$ (round metric)

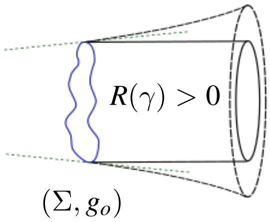
Adapted Mantoulidis–Schoen (for ($\star$))

Construct a collar extension



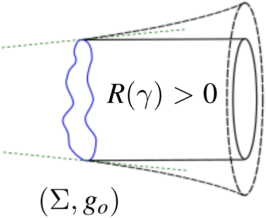
Adapted Mantoulidis–Schoen (for $(\star)$)

Construct a collar extension



Adapted Mantoulidis–Schoen (for $(\star)$)

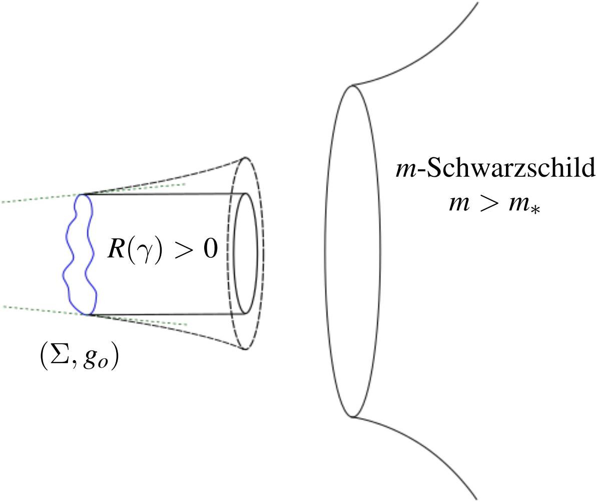
Construct a collar extension



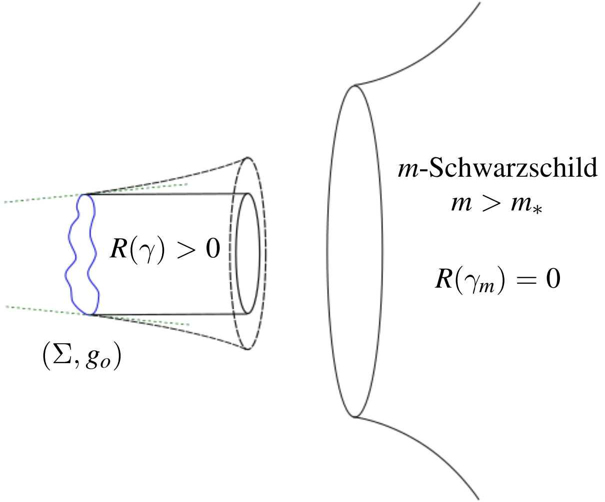
Estimate $\mathfrak{m}_H(\Sigma_1)$ (at the boundary).

Say, $m_* \geq \mathfrak{m}_H(\Sigma)$.

Adapted Mantoulidis–Schoen (for $(\star)$)



Adapted Mantoulidis–Schoen (for $(\star)$)



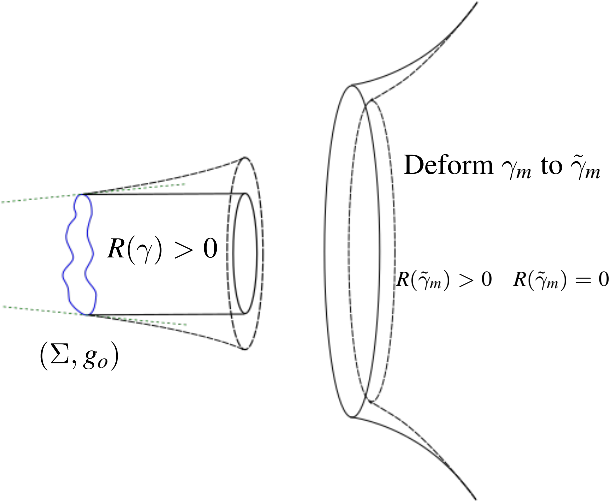
Mantoulidis–Schoen developed an analytic tool to glue two rotationally symmetric manifolds satisfying certain geometric conditions. Then, by picking the deformation parameter and the collar extension appropriately, they can glue the collar to a Schwarzschild manifold via a positive scalar bridge.

Proposition (C., Cederbaum, McCormick and Miao, 2017). Consider a metric $\gamma_f := ds^2 + f(s)^2 g_*$ on $[a,b] \times \mathbb{S}^2$, where f is a smooth, positive, and increasing function on $[a,b]$. Suppose that

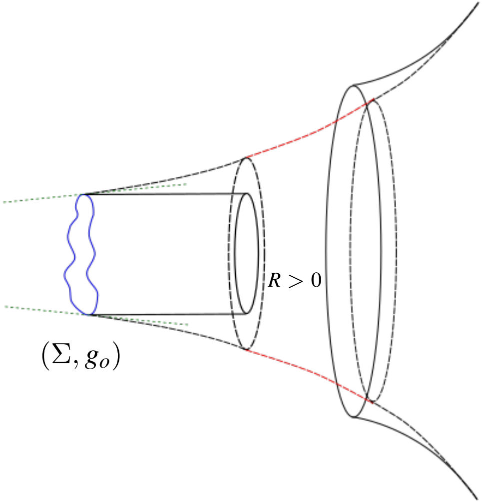
- γ_f has positive scalar curvature,
- $\Sigma_b = \{b\} \times \mathbb{S}^2$ has positive mean curvature, and
- $\mathfrak{m}_H(\Sigma_b) \geq 0$.

Then, for any $m > \mathfrak{m}_H(\Sigma_b)$, the manifold $([a,b] \times \mathbb{S}^2, \gamma_f)$ can be smoothly glued to (an exterior region of) a Schwarzschild manifold of mass m.

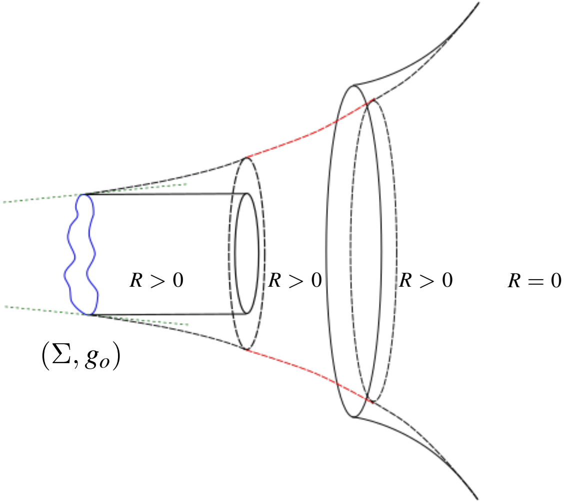
Adapted Mantoulidis–Schoen (for $(\star)$)



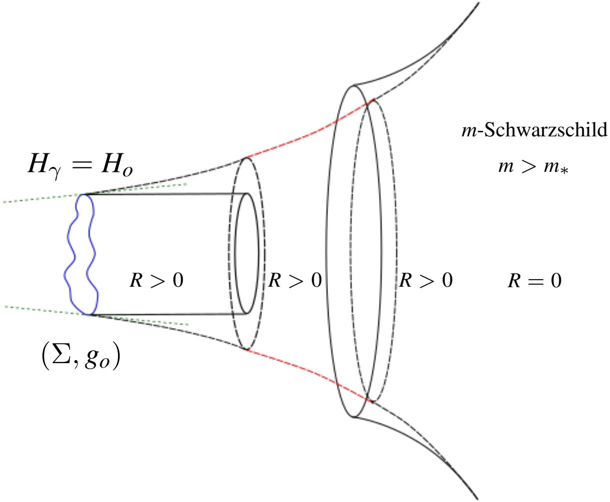
Adapted Mantoulidis–Schoen (for $(\star)$)



Adapted Mantoulidis–Schoen (for $(\star)$)



Adapted Mantoulidis–Schoen (for $(\star)$)



How to obtain $\{g(t)\}_{0\leq t\leq 1}$?

By uniformization $g_o = e^{2w}g_*$, where w is a smooth function on $\Sigma \cong \mathbb{S}^2$. Let $\tilde{g}(t) = e^{2w(1-t)}g_*$. By a reparametrization and by modifying this path via a family of diffeomorphisms on $\mathbb{S}^2$, we can get the desired path (Mantoulidis–Schoen).

For higher dimensions **(C.–Miao, 2016)**: For $n = 3$, smoothing out the a continuous path obtained with Ricci flow with surgeries my Marques. For $n \geq 4$, use an type of inverse curvature flow studied by Gerhardt and Urbas.

Collar extensions:

- For $H_o \equiv 0$ (Mantoulidis–Schoen), it is enough to control the area growth. For $\varepsilon > 0$

$$\gamma = A_o^2 u(t,\cdot)^2 dt^2 + (1+\varepsilon t^2)^{1/2} g(t),$$

  with A_o sufficiently big so that $R(\gamma) > 0$. This also makes the boundary outer-minimizing.

- For $H_o > 0$, we use collar extensions by Miao and Xie, given by

$$\gamma = A_o^2 dt^2 + F(t)^2 g(t),$$

where A_o sufficiently big and F is a type of "Schwarzschildean necks".

The collar extensions constructed by Miao–Xie have well-controlled Hawking mass, and give

$$\mathfrak{m}_H(\Sigma_1) \leq \left[1 + \left(\frac{\alpha \frac{H_o^2 r_o^2}{4}}{\beta - (1+\alpha)\frac{H_o^2 r_o^2}{4}}\right)^{1/2}\right] \mathfrak{m}_H(\Sigma, g_o, H_o).$$

Theorem ($\star$) (C., Cederbaum, McCormick and Miao, 2017). Given CMC Bartnik data ($H_o > 0$ constant) $(\Sigma \cong \mathbb{S}^2, g_o, H_o)$ with $K(g_o) > 0$, there exist constants $\alpha \geq 0$ and $0 < \beta \leq 1$ (depending on g_o: $\alpha = 0$ and $\beta = 1$ iff g_o is round) such that if

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then,

$$\mathfrak{m}_B(\Sigma, g_o, H_o) \leq \left[1 + \left(\frac{\alpha \frac{H_o^2 r_o^2}{4}}{\beta - (1+\alpha)\frac{H_o^2 r_o^2}{4}}\right)^{1/2}\right] \mathfrak{m}_H(\Sigma, g_o, H_o).$$

Theorem (Miao, Wang and Xie, 2018).
Let $(\Sigma \cong \mathbb{S}^2, g_o, H_o)$ be CMC Bartnik data with $K(g_o) > 0$, with $\frac{H_o r_o}{2} \leq 1$. Then

$$\mathfrak{m}_B(\Sigma \cong \mathbb{S}^2, g_o, H_o) \leq \frac{3}{4} r_o (1 + \zeta_{g_o} H_o r_o) \zeta_{g_o} H_o r_o + \mathfrak{m}_H(\Sigma, g_o, H_o),$$

where $\zeta_{g_o} = \inf \left(\frac{\alpha}{2\beta} \right)^{1/2}$, and the infimum is taken over the set of paths $\{g(t)\}_{0 \leq t \leq 1}$ with $K(g(t)) > 0$ connecting g_o to a round metric and satisfying $\mathrm{tr}_{g(t)} g'(t) \equiv 0$ for all $t \in [0, 1]$.

When the value of H_o is small, this estimate improves the estimate in Theorem $(\star)$.

Thank you!

And thank you, Greg, for all your support, encouragement and friendship. Happy birthday!