Bakry-Émery curvature-dimension conditions in relativity

Eric Woolgar

Dept of Mathematical and Statistical Sciences
University of Alberta
ewoolgar@math.ualberta.ca
http://www.math.ualberta.ca/~ewoolgar

December 2018

Black hole topology I

Hawking's theorem:

- Hawking 1972: 4-dimensions, dominant energy $\implies$ S^2 apparent horizon.
- Gibbons, EW 1999: Area of horizon $\searrow 0$ as energy condition violation $\nearrow 0$.
- Galloway-Schoen 2006: Higher dimensions $\implies$ positive Yamabe type except in special case; stability argument for apparent horizons.
- Galloway 2008: Removed the special case.

Black hole topology II

Topological Censorship:

- Gannon 1975: Topology on a Cauchy surface $\implies$ singularity in future.
- Friedman-Schleich-Witt 1993: "Horizons censor the topology".
- Chruściel-Wald 1994: Application to Black holes.
- Galloway 1995: DOC simply connected.
- Galloway-Schleich-Witt-EW 1999: Topcen in AdS, genus formula.
- Eichmair-Galloway-Pollack 2013: Initial data formulation: Outside all MOTSs, topology is simple.

Black hole topology III

Extreme (degenerate) vacuum horizons:

Theorem(Khuri-EW-Wylie): Let $\mathcal{H}$ be a (cross-section of a) degenerate horizon of a D-dimensional stationary vacuum black hole spacetime with cosmological constant $\Lambda \geq 0$.

- $\Lambda > 0 \implies \pi_1(\mathcal{H}) < \infty$.
- $\Lambda = 0 \implies \pi_1(\mathcal{H})$ contains a finite-index Abelian subgroup $\simeq \mathcal{Z}^k$ with $k \leq D - 4$; indeed, $b_1(\mathcal{H}) \leq D - 4$.

In short, in any dimension, the *Universal Covering Space* will be either

- a compact manifold if $\Lambda > 0$, or medskip
- a product of a compact manifold with $\mathbb{E}^k$ if $\Lambda = 0$.

Near Horizon Geometries (NHG)

Extreme (cold, degenerate) black holes:

- Killing horizon $\mathcal{H}$ with KVF $\xi = \frac{\partial}{\partial v}$.
- Zero surface gravity: $\nabla_\xi \xi\big|_{\mathcal{H}} = \kappa \xi$, $\kappa = 0$.
- Normal coordinates: $\mathcal{H} := \{r = 0\}$, $0 < C \leq F(r, x)$:

$$ds^2 = 2dv\left(dr - \frac{1}{2}r^2 F(r,x)dv - rX_a(r,x)dx^a\right) + g_{ab}(r,x)dx^a dx^b$$

- Then replace $v \mapsto v/\epsilon$, $r \mapsto \epsilon r$, $\epsilon \searrow 0$:

$$\mathrm{Ric}(g) + \frac{1}{2}\pounds_X g - \frac{1}{2}X \otimes X = \Lambda g + \text{ matter } \geq \Lambda g$$

and

$$F = \frac{1}{2}|X|_g^2 - \frac{1}{2}\operatorname{div}_g X + \Lambda + \text{ matter}$$

Near Horizon Geometries (NHG)

Extreme (cold, degenerate) black holes:

- Killing horizon $\mathcal{H}$ with KVF $\xi = \frac{\partial}{\partial v}$.
- Zero surface gravity: $\nabla_\xi \xi\big|_{\mathcal{H}} = \kappa\xi$, $\kappa = 0$.
- Normal coordinates: $\mathcal{H} := \{r = 0\}$, $0 < C \le F(r, x)$:

$$ds^2 = 2dv\left(dr - \frac{1}{2}r^2F(r,x)dv - rX_a(r,x)dx^a\right) + g_{ab}(r,x)dx^a dx^b$$

- Then replace $v \mapsto v/\epsilon$, $r \mapsto \epsilon r$, $\epsilon \searrow 0$:

$$\mathrm{Ric}(g) + \frac{1}{2}\pounds_X g - \frac{1}{2}X \otimes X = \Lambda g + \text{ matter } \ge \Lambda g$$

and

$$F = \frac{1}{2}|X|_g^2 - \frac{1}{2}\operatorname{div}_g X + \Lambda + \text{ matter}$$

Bakry-Émery-Ricci tensor

$$\begin{aligned} \mathrm{Ric}_X^N(g) &:= \mathrm{Ric}(g) + \frac{1}{2}\pounds_X g - \frac{1}{N-n} X \otimes X \ , \\ \mathrm{Ric}_X^{\pm\infty}(g) &:= \mathrm{Ric}(g) + \frac{1}{2}\pounds_X g \ . \end{aligned} \tag{1}$$

Notation:

- $n =$ dimension of manifold
- $N =$ *synthetic dimension*. (Some use this term for $m = N - n$).
- *Positive*: $N \in (n, \infty)$
- *Negative*: $N \in [-\infty, 1]$: Identify $N = \infty$ and $N = -\infty$ cases.

Synthetic dimension: Kaluza-Klein/Warped products

- Warped product $\mathcal{N}^N = \mathcal{M}^n \times_{\varepsilon e^{-f/(N-n)}} \mathcal{F}$

$$g_{\mathcal{N}} = g_{\mathcal{M}} \oplus \varepsilon^2 e^{-2f/(N-n)} g_{\mathcal{F}}$$

- Then

$$\begin{aligned}\mathrm{Ric}(g_{\mathcal{N}}) = &\left[\mathrm{Ric}(g_{\mathcal{M}}) + \mathrm{Hess}_{g_{\mathcal{M}}} f - \frac{1}{(N-n)} df \otimes df\right] \\ &\oplus \left[\mathrm{Ric}(g_{\mathcal{F}}) + \frac{\varepsilon^2}{(N-n)} e^{-2f/(N-n)} g_{\mathcal{F}} \left(\Delta_{g_{\mathcal{M}}} f - |df|_g^2\right)\right] .\end{aligned}$$

- Justifies the term *synthetic dimension* in gradient $X = df$ case.

Synthetic dimension: Kaluza-Klein/Warped products

- Warped product $\mathcal{N}^N = \mathcal{M}^n \times_{\varepsilon e^{-f/(N-n)}} \mathcal{F}$

$$g_{\mathcal{N}} = g_{\mathcal{M}} \oplus \varepsilon^2 e^{-2f/(N-n)} g_{\mathcal{F}}$$

- Then

$$\begin{aligned} \mathrm{Ric}(g_{\mathcal{N}}) = & \left[\mathrm{Ric}(g_{\mathcal{M}}) + \mathrm{Hess}_{g_{\mathcal{M}}} f - \frac{1}{(N-n)} df \otimes df \right] \\ & \oplus \left[\mathrm{Ric}(g_{\mathcal{F}}) + \frac{\varepsilon^2}{(N-n)} e^{-2f/(N-n)} g_{\mathcal{F}} \left(\Delta_{g_{\mathcal{M}}} f - |df|_g^2 \right) \right] . \end{aligned}$$

- Justifies the term *synthetic dimension* in gradient $X = df$ case.

Bakry-Émery tensor in physics

- Scalar-tensor theory: $X = df$, N arbitrary (including $N < 0$).
- Static Einstein: $X = df$, $N = n + 1$.
- Optical metric for static Einstein: $X = df$, $N = 1$.
- Kaluza-Klein dilaton: $X = df$, $N = n + k$.
- Near-horizon geometries: $N = n + 2$ (arbitrary X).
- Yang-Mills energy gap: $X = df$, $N = \infty$. (Lichnérowicz, Moncrief-Marini-Maitra arxiv:1809.06318)

Riemannian Bakry-Émery: Degenerate horizon topology

Structure Theorem (Khuri-EW-Wylie)

Suppose $(\mathcal{M}, g)$ is a compact (complete) Riemannian manifold with $\mathrm{Ric}_X^N(\mathcal{M}) \geq 0$ for some $N \in [n+1, \infty)$. Then

- the universal cover splits isometrically as a product $\mathcal{N} \times \mathbb{R}^p$ where $\mathcal{N}$ is compact (complete),
- $\mathrm{Ric}_X^N(\mathcal{N}) \geq 0$, and
- X is tangent to $\mathcal{N}$.

Main estimate for the proof

- As in Cheeger-Gromoll, the Laplacian of the distance function ρ from some point, computed at another point lying along a minimal geodesic γ joining the two points, obeys

$$\Delta\rho \leq \int_0^\rho \left[\frac{(n-1)}{\rho^2} - \frac{t^2}{\rho^2}\,\mathrm{Ric}(\dot\gamma,\dot\gamma)\right] dt \ .$$

- Apply $\mathrm{Ric}_X^N \geq 0$, complete a square, get

$$\Delta\rho \leq \frac{N-1}{\rho} + \nabla_X\rho \ .$$

- Defining $\Delta_X\rho := \Delta\rho - \nabla_X\rho$, then

$$\Delta_X\rho \leq \frac{N-1}{\rho} \to 0 \text{ as } \rho\to\infty \ .$$

- Differences from Cheeger-Gromoll: $n \mapsto N$, $\Delta\rho \mapsto \Delta_X\rho$.

Main estimate continued

- Apply to Busemann (support) functions

$$b^{\gamma}(q) := \lim_{t\to\infty} [t - \mathrm{dist}(q, \gamma(t))]$$

- $\Delta_X b^{\pm} \geq 0$ as $\rho \to \infty$.
- A triangle inequality argument implies then that $\Delta_X b^{\pm} = 0$.

Sketch the proof

- One also obtains (by direct manipulation)

$$\Delta_X\left(|\nabla u|^2\right)=2|\operatorname{Hess} u|^2+2\nabla_{\nabla u}\left(\Delta_X u\right)+2\operatorname{Ric}_X^m(\nabla u,\nabla u)+\frac{2}{m}\left[X(u)\right]^2 .$$

- Now apply these results to Busemann functions $u=b^\pm$ defined by line γ.
- Putting everything together, get

$$0=\Delta_X\left(|\nabla b^\pm|^2\right)\geq 2|\operatorname{Hess} b^\pm|^2\geq 0.$$

- Hence $b^\pm$ are linear: their level sets define the splitting.
- Repeat until there are no more lines to split off.

Lorentzian Bakry-Émery

Lorentzian examples

- Singularity theorems: Often treated as "uniquely natural" predictions of general relativity.
- But are they just as natural in less geometrical settings; e.g., scalar-tensor gravitation?
- JS Case (2010): Hawking-Penrose-type theorem with $X = df$.
- GJ Galloway and EW: Cosmological singularity and splitting theorems with $N = \infty$, $X = df$
- EW and Will Wylie (2015, 2018): general N, splitting theorems, still with $X = df$.

Causal curvature-dimension conditions

- Fix some $N \in \mathbb{R} \cup \{\infty\}$, $\lambda \in \mathbb{R}$.
- The *timelike curvature-dimension condition* $\mathrm{TCD}(\lambda, N)$ is

$$\mathrm{Ric}_f^N(X,X) \geq \lambda \in \mathbb{R}$$

  for every unit timelike vector X.
- The *null curvature-dimension condition* $\mathrm{NCD}(N)$ is

$$\mathrm{Ric}_f^N(X,X) \geq 0 \in \mathbb{R}$$

  for every null vector X.
- These reduce to $\mathrm{Ric}(X,X) \geq 0$ if f is constant.
- In general relativity:
  - $\mathrm{Ric}(X,X) \geq 0$ follows from the *strong energy condition*.

Typical conditions on f when $N = \infty$ or $N \leq 1$

These conditions are only needed when $N = \infty$ or $N \leq 1$ (or $N \leq 2$ for certain Lorentzian theorems).

(a) The "classic" condition: $f \leq k$.

(b) Wylie's f-completeness condition: $\int_0^\infty e^{-2f(t)/(n-1)} dt = \infty$ along (certain) complete geodesics.[1]

(c) Sometimes need a stronger condition: ∇f future-timelike to the future of a Cauchy surface S.

[1] $f(t)$ is short-hand for $f \circ \gamma(t)$ where γ is a geodesic.

E.g.: Hawking-type cosmological singularity theorem

(GJ Galloway and EW for $N = \pm\infty$; W Wylie and EW for general N). Assume that

- $\mathrm{TCD}(0, N)$ holds for some fixed $N \in [-\infty, 1] \cup (n, \infty]$,
- S is a compact Cauchy surface, ν its future unit normal,
- the (future) f-mean curvature of S obeys $H_f := H - \nabla_\nu f < 0$ everywhere on S, and
- if $N \in [-\infty, 1]$ then $\int_0^\infty e^{-2f(s)/(n-1)} ds$ diverges along every complete timelike geodesic orthogonal to S.

Then no timelike geodesic is future-complete.

TCD Condition

- Recall $\mathrm{TCD}(0,N) \Rightarrow \mathrm{Ric}(X,X) + \mathrm{Hess}(X,X)f - \frac{1}{(N-n)}\langle df, X\rangle^2 \geq 0$.
- When $N > n$, the $\langle df, X\rangle^2$ term *helps*: no control of f required.
- When $N \leq 1$, the $\langle df, X\rangle^2$ term *hinders*: but can still obtain a theorem if we have mild control of f.
- No results for $N \in (1, n]$.

Related splitting theorem

Assume that

- TCD$(0, N)$ holds for some fixed $N \in [-\infty, 1] \cup (n, \infty]$,
- S is a compact Cauchy surface, ν its future unit normal,
- the (future) f-mean curvature of S obeys $H_f := H - \nabla_\nu f \leq 0$ everywhere,
- if $N \in [-\infty, 1]$ then $\int_0^\infty e^{-2f(s)/(n-1)} ds$ diverges along every complete timelike geodesic orthogonal to S, and
- the geodesics orthogonal to S are future-complete.

Then,

- if $N \in (-\infty, 1) \cup (n, \infty]$, the future of S is isometric to $-dt^2 \oplus h$ and f is independent of t (answers question of JS Case).
- if $N = 1$, the future of S is isometric to $-dt^2 \oplus e^{2\psi(t)/(n-1)} h$ and $f = \psi(t) + \phi(y)$, $y \in S$.

The (timelike) f-Raychaudhuri equation

$$\frac{\partial H}{\partial t} = -\operatorname{Ric}(\gamma', \gamma') - |K|^2 = -\operatorname{Ric}(\gamma', \gamma') - |\sigma|^2 - \frac{H^2}{(n-1)}$$

Use $H_f := H - f'$ and use definition of Ric_f^N. Get

$$\begin{aligned} \frac{\partial H_f}{\partial t} &= -\operatorname{Ric}_f^N(\gamma', \gamma') - |\sigma|^2 - \frac{H^2}{(n-1)} - \frac{f'^2}{(N-n)} \\ &= -\operatorname{Ric}_f^N(\gamma', \gamma') - |\sigma|^2 - \frac{1}{(n-1)} \left[H_f^2 + 2H_f f' + \frac{(N-1)}{(N-n)} f'^2 \right] \end{aligned}$$

Analyse this. Use that H_f diverges along γ at finite t iff H diverges.

- First line: If $N > n$ each term on right is ≤ 0 (assuming $\mathrm{TCD}(0, N)$).
- Second line: Coefficient of f'^2 has same sign for $N < 1$ as for $N > n$, but must deal with $H_f f'$ term.

Focusing argument: TCD$(0, N)$ case

- For $N > n$, an easy identity yields

$$\frac{\partial H_f}{\partial t} \leq -\operatorname{Ric}_f^N(\gamma',\gamma') - |\sigma|^2 - \frac{H_f^2}{(N-1)}$$

$$\Rightarrow \frac{\partial x}{\partial t} \leq -x^2\ ,\ x := H_f/(N-1)\ ,\ \text{using TCD}(0,N)\ .$$

- Otherwise, use an integrating factor to eliminate $H_f f'$ term:

$$\frac{\partial}{\partial t}\left(e^{\frac{2f}{(n-1)}} H_f\right) = -e^{\frac{2f}{(n-1)}}\left[\operatorname{Ric}_f^N(\gamma',\gamma') + |\sigma|^2 + H_f^2 + \frac{(N-1)f'^2}{(N-n)(n-1)}\right]$$

$$\Rightarrow \frac{\partial x}{\partial t} \leq -e^{-\frac{2f}{(n-1)}} x^2\ ,\ x := e^{\frac{2f}{(n-1)}} H_f\ ,\ \text{using TCD}(0,N)\ .$$

- Now $x(0) \leq x_0 < 0$. $\begin{cases} x(t) \leq \frac{1}{t+1/x_0}, & N > n \\ x(t) \leq \frac{1}{\int_0^t e^{-2f(s)/(n-1)} ds + 1/x_0}, & N \in [-\infty, 1] \end{cases}$
- Thus $x(t) \to -\infty$ as $t \nearrow t_0$.

Completion of the argument.

- $x \to -\infty$ as $t \to t_0$ for some $t_0 \leq T(x_0) \leq T$.
- Thus $H \to -\infty$ as $t \to t_0$ for some $t_0 \leq T(x_0) \leq T$.
- Thus no future-timelike geodesic orthogonal to S can maximize beyond $t = T$.
- If there were a future-complete timelike geodesic γ, there would be a sequence of maximizing geodesics from S to γ, meeting S orthogonally and of unbounded length.
- Thus there can be no future-complete timelike geodesic. QED.

Splitting argument

- Now $H_f \leq 0$, and we assume future completeness.
- If $H_f < 0$ on S, cannot be future complete, so $H_f = 0$ at least somewhere on S.
- If H_f is not *identically* zero on S, do short f-mean curvature flow.

$$\frac{\partial X}{\partial s} = -H_f \nu \ .$$

- Strong maximum principle implies that $H_f(s) < 0$ for $s > 0$ (and still Cauchy).
- Therefore must have $H_f \equiv 0$ on S.
- And must have $H_f(t) \equiv 0$, so each term on right in Raychaudhuri equation must vanish.

Splitting argument: continued

- For $N > n$, recall

$$\frac{\partial H_f}{\partial t} = -\operatorname{Ric}_f^N(\gamma', \gamma') - |\sigma|^2 - \frac{H^2}{(n-1)} - \frac{f'^2}{(N-n)} \; .$$

- Must have $H_f \equiv 0$ on $(0, t)$.
- Thus $\sigma = 0$, $H = 0$, $f' = 0$ on $(0, t)$.
- $g = -dt^2 \oplus h$, $f' = 0$, and since the γ are future-complete, the splitting is global.

Splitting argument: continued

- For $N \in [-\infty, 1]$, had

$$\frac{\partial}{\partial t}\left(e^{\frac{2f}{(n-1)}} H_f\right) = -\, e^{\frac{2f}{(n-1)}}\left[\mathrm{Ric}_f^N(\gamma', \gamma') + |\sigma|^2 \right.$$
$$\left. + H_f^2 + \frac{(N-1)f'^2}{(N-n)(n-1)}\right]$$

- Must have $H_f \equiv 0$ on $(0, t)$.
- Thus $\sigma = 0$, $H = f'$, and *either* $f' = 0$ or $N = 1$, on $(0, t)$.
- If $N \neq 1$, get $H = 0$ and get global product splitting as before.
- If $N = 1$, use also that $\mathrm{Ric}_f^1(\gamma', \gamma') = 0$ on $(0, t)$.
- A computation then yields the warped product of the theorem.

A Myers theorem version of Hawking's theorem

- Let
  - (M, g) admit a compact Cauchy surface S, and
  - $\mathrm{Ric}(t, t) \geq k > 0$ for every timelike vector t^a.
- Then $\mathrm{vol}(M) \leq \frac{2\pi}{k} \mathrm{vol}(S)$.

Makes contact with approaches to Ricci curvature lower bounds in metric-measure context, where Myers's theorem becomes a statement about the support of a measure.

Ricci curvature lower bounds (McCann 1808.01536)

- Lorentzian version of Lott-Villani, Sturm, etc.
- Measures on spacetime: $d\,\mathrm{vol}_g$, $dm := e^{-f}d\,\mathrm{vol}_g$, $d\mu_s$, with $\rho := \frac{d\mu_s}{dm}$.
- Entropy $e(s) := E_f(\mu_s) := \int_M \rho \log \rho dm$.
- Choose $[0,1] \ni s \mapsto \mu_s$ to be a "displacement interpolant" curve for an optimal transport with cost $\ell_q(\mu_0, \mu_1) = \sup\left((\ell(x,y))^q d\pi\right)^{1/q}$, where π is a "coupling" of μ_0 to μ_1, the supremum is over all couplings π, and $\ell(x,y)$ is Lorentzian distance between points, defined to be $-\infty$ if the points are not timelike-related.
- Main result: For smooth Riemannian manifolds

$$\mathrm{TCD}(0,N) \Leftrightarrow e''(s) \geq \frac{1}{N}\left(e'(s)\right)^2 + K\left(\ell_q(\mu_0,\mu_1)\right)^2$$

  along a displacement interpolant.

Aim: to generalize energy conditions to weak setting.