Using timelike completeness to prove inextendibility of spacetimes in low regularity

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GregFest!

Joint work with Greg Galloway, Jan Sbierski, and Melanie Graf.

Motivation

Strong Cosmic Censorship Conjecture:

The maximal globally hyperbolic development of generic initial data for Einstein's equations is inextendible as a suitably regular Lorentzian manifold.

Introduction

- This talk will mostly be concerned with the question:

$$\boxed{\text{Which spacetimes are } C^0\text{-inextendible?}}$$

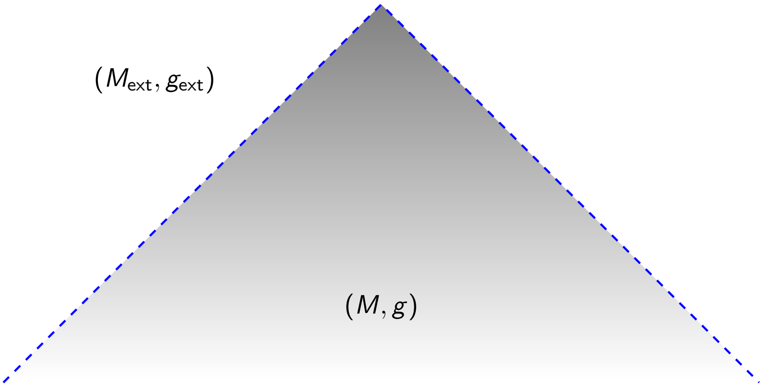
- A spacetime (M, g) is *C^0-extendible* if there is a spacetime $(M_{\mathrm{ext}}, g_{\mathrm{ext}})$ with a C^0 metric such that (M, g) embeds

  $$(M, g) \hookrightarrow (M_{\mathrm{ext}}, g_{\mathrm{ext}})$$

  properly and isometrically.

- If no such extension exists, then (M, g) is *C^0-inextendible*.

Introduction



Motivation

Jan Sbierski showed

Theorem (Sbierski (2015))

The maximally analytic Schwarzschild spacetime is C^0-inextendible.

In his paper he raised an open question:

Are timelike complete spacetimes C^0-inextendible?

Answer to Sbierski's Question

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We don't know.

But we do know...

But we do know...

Theorem (Galloway, L., and Sbierski (2017))

A globally hyperbolic and timelike complete spacetime is C^0-inextendible.

Theorem (Graf and L. (2017))

A timelike complete spacetime is inextendible as a spacetime with a Lipschitz metric (i.e. it's $C^{0,1}$-inextendible).

Future and Past Boundaries

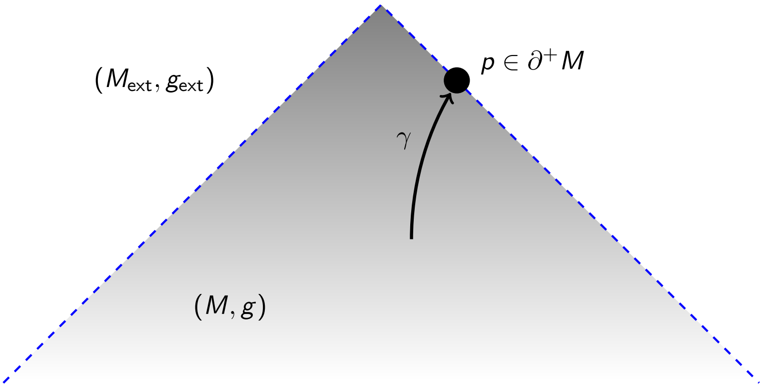
Suppose $(M_{\text{ext}}, g_{\text{ext}})$ is a C^0-extension of (M, g).

- The *future boundary of M*, denoted by $\partial^+ M$, is the set of points $p \in \partial M$ such that there is a smooth future directed timelike curve $\gamma : [0,1] \to M_{\text{ext}}$ satisfying

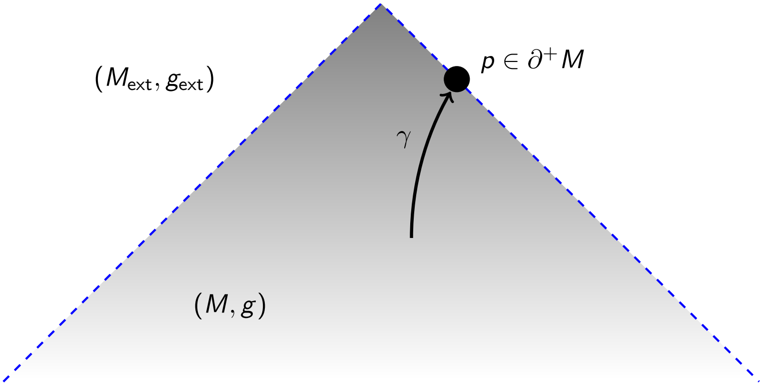
$$\gamma(1) = p \quad \text{and} \quad \gamma\big([0,1)\big) \subset M.$$

- The *past boundary of M*, denoted by $\partial^- M$, is defined time dually.

$p \in \partial^+ M$

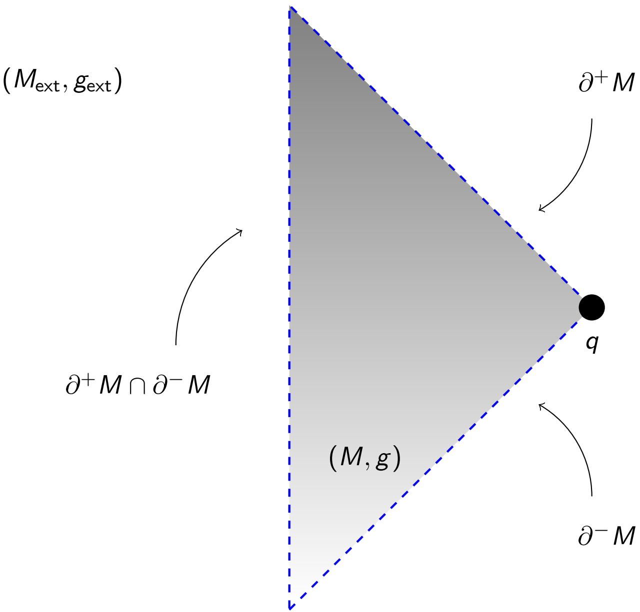


$p \in \partial^+ M$



$$\gamma(1) = p \quad \text{and} \quad \gamma\big([0,1)\big) \subset M.$$

Various points on $\partial^+ M$ and $\partial^- M$



Lemma for C^0-inextendibility

Lemma (Sbierski)

Suppose (M_{ext}, g_{ext}) is a C^0-extension of (M, g). Then

$$\partial^+ M \cup \partial^- M \neq \emptyset.$$

If one assumes a C^0-extension of (M, g) and proves

$$\partial^+ M = \emptyset \quad \text{and} \quad \partial^- M = \emptyset,$$

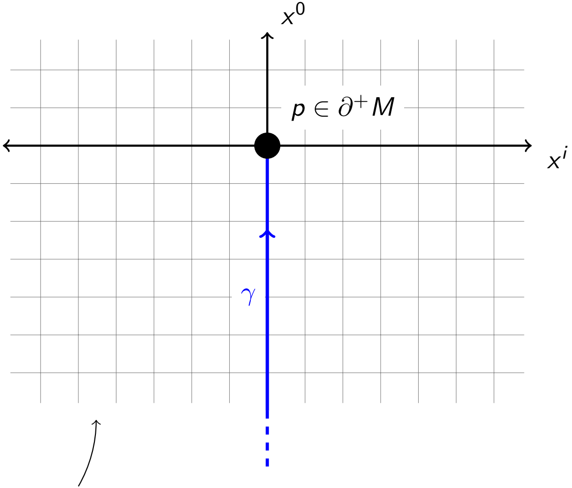
then the Lemma yields a contradiction. Thus (M, g) is C^0-inextendible.

Plan of attack

Plan of attack

Somehow use future timelike completeness to show $\partial^+ M = \emptyset$.

A neighborhood about $p \in \partial^+ M$



A small neighborhood about p.

What do we know about this neighborhood?

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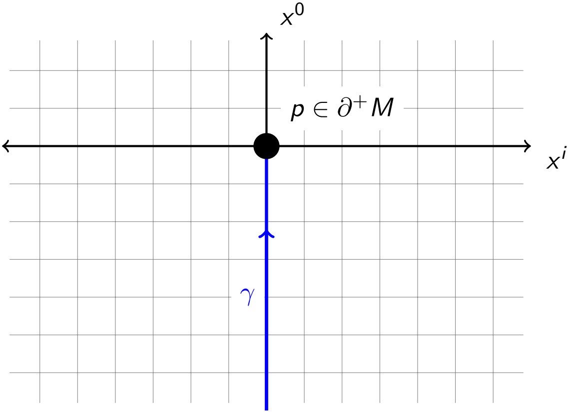


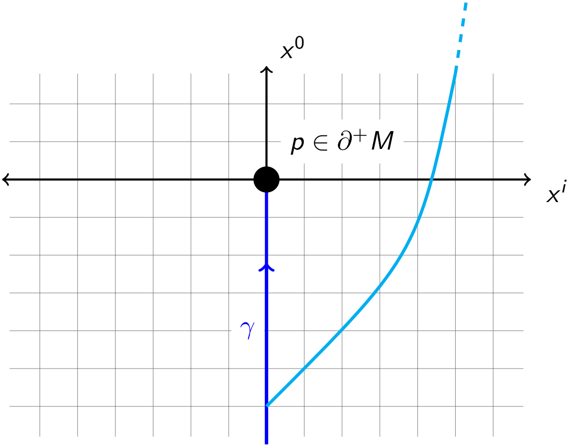
- $g_{\mu\nu}(p) = \eta_{\mu\nu}$
- $|g_{\mu\nu}(x) - \eta_{\mu\nu}| < \varepsilon$
- The negative x^0-axis makes up γ which lies in M.

Can we find a timelike a geodesic?

Can we find a timelike a geodesic?

Sure we can!

A timelike geodesic



We found one! Does it do us any good? Nope.

How do we proceed?

How do we proceed?

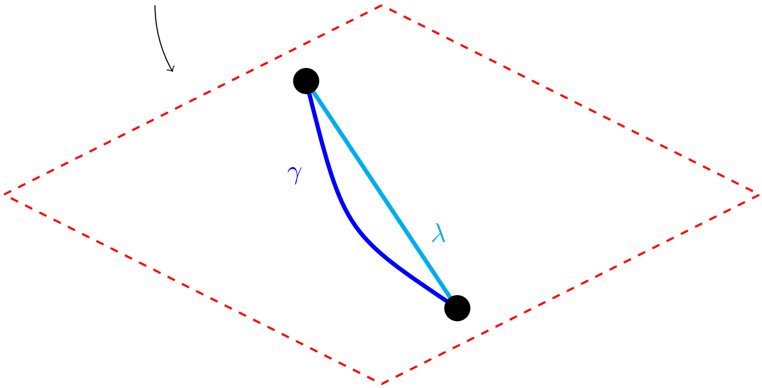
We need to find an invariant quantity.

An invariant quantity

Globally hyperbolic spacetimes have causal maximizers.

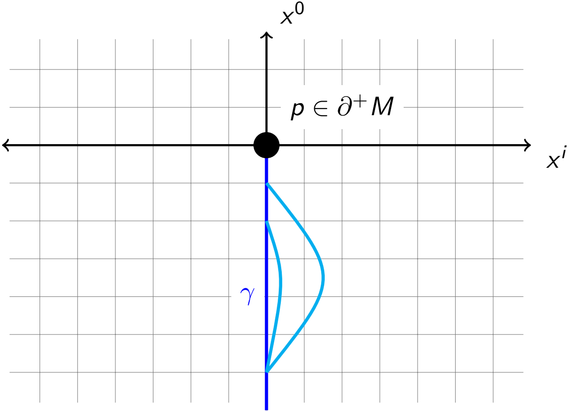
A causal maximizer

A globally hyperbolic spacetime.

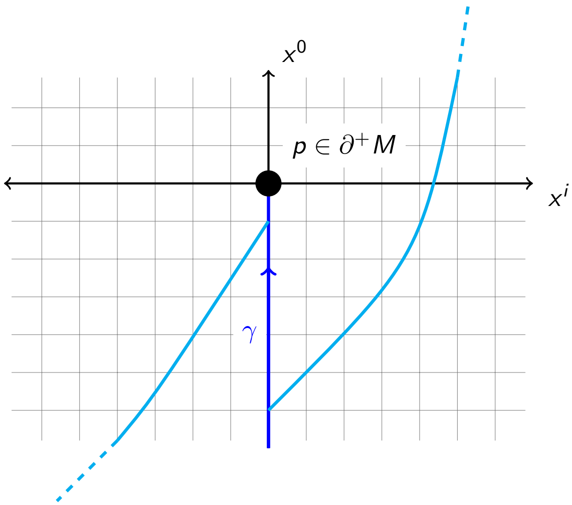


$L(\lambda) \geq L(\gamma)$

Back to our neighborhood



Causal maximizers can leave our neighborhood



We want to keep the focus within our neighborhood

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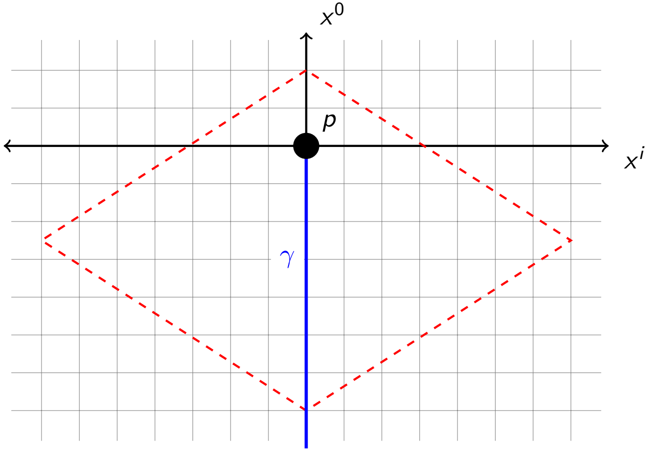
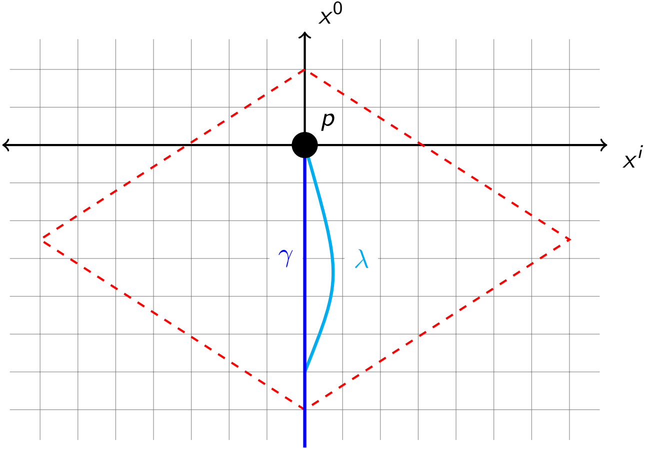
We take a globally hyperbolic subset of our neighborhood.

A globally hyperbolic subset of our neighborhood



Finding a causal maximizer



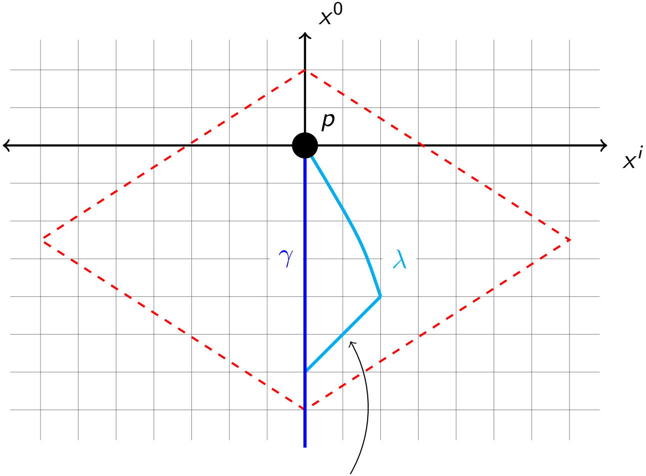
$L(\lambda) \geq L(\gamma)$

The causal maximizer λ

- $\lambda|_M$ is future inextendible in M.
- λ is a causal maximizer.
- M is timelike complete.

Therefore the portion of λ lying in M must be a null geodesic.

λ is null within M



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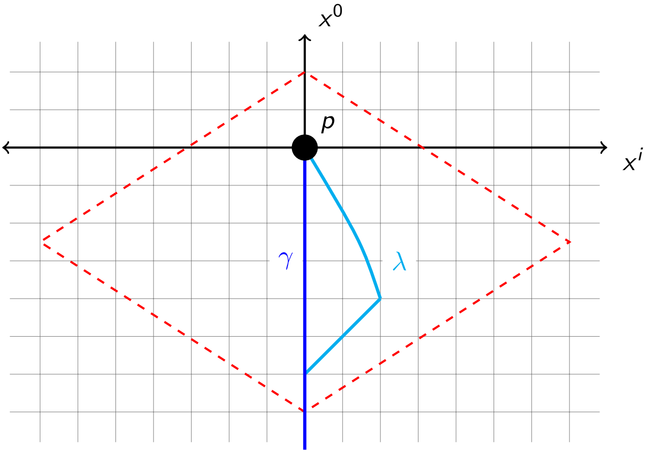

$L(\lambda) \geq L(\gamma)$

This picture seems to violate the twin paradox.

Interlude: the twin paradox

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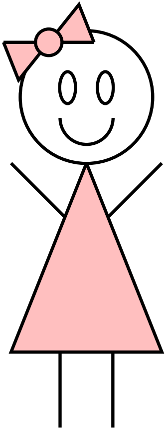
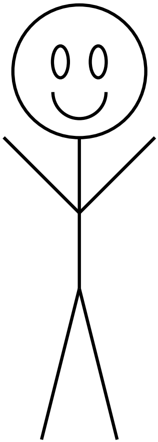
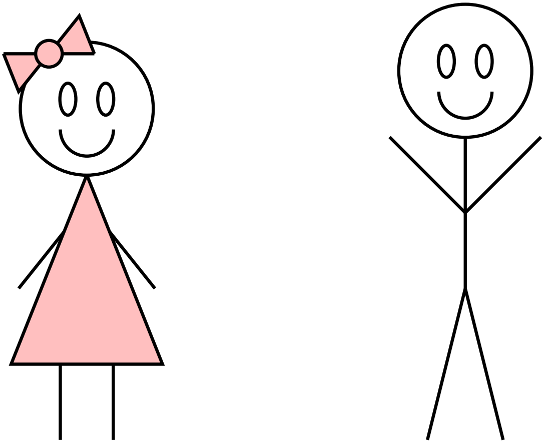


Alice



Bob

Interlude: the twin paradox



Bob was born just 2 minutes before Alice.

Interlude: the twin paradox

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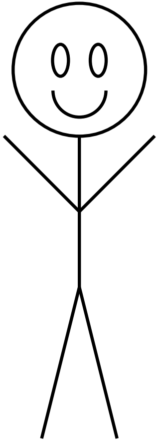
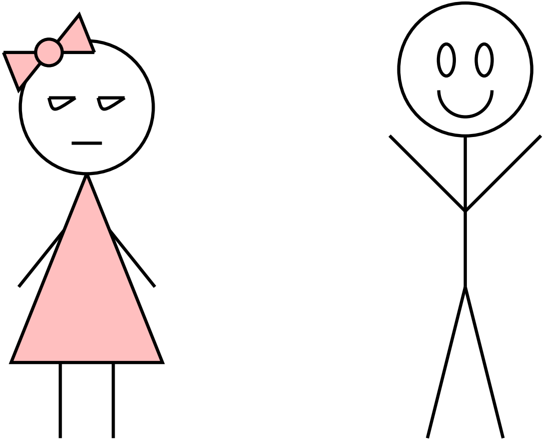
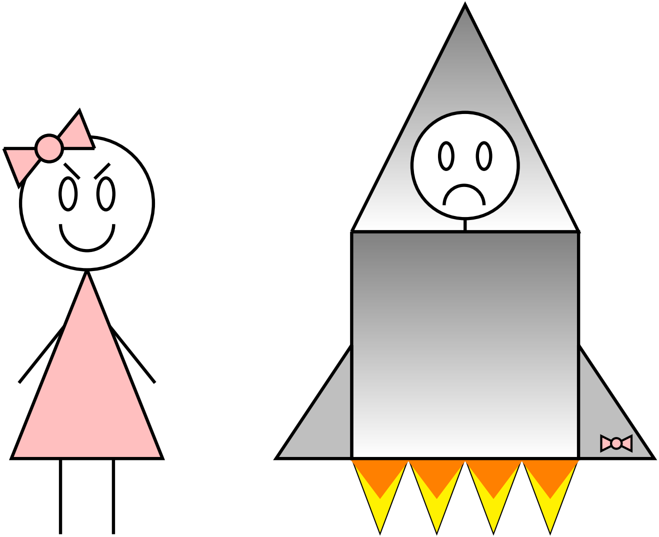


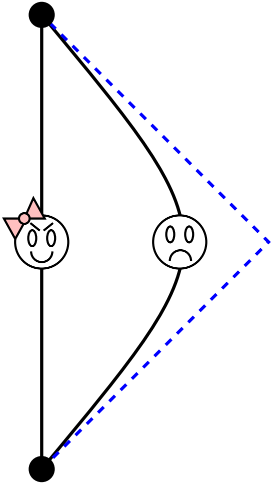
Interlude: the twin paradox



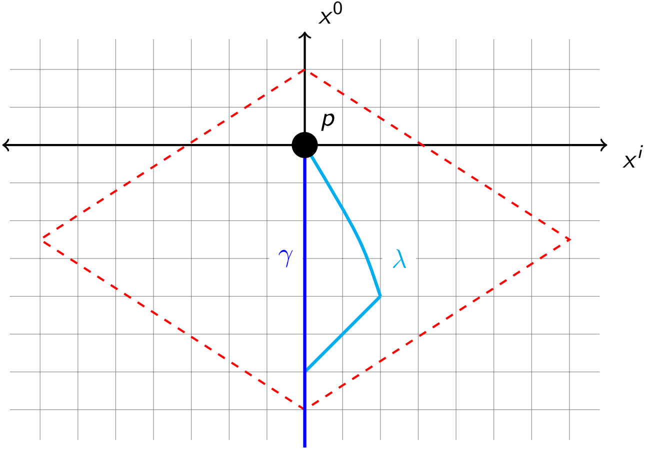
Interlude: the twin paradox



Interlude: the twin paradox



Back to our neighborhood

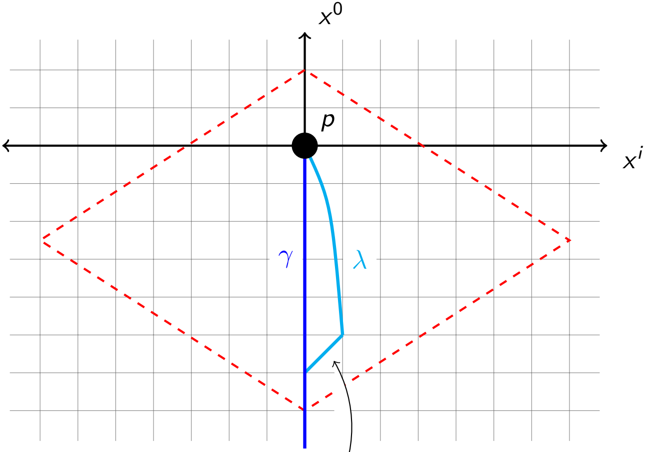


$L(\lambda) \geq L(\gamma)$

A problem

We don't know how much of λ is in M.

A problem



$L(\lambda) \geq L(\gamma)$

Only a small portion of λ is in M.

Global hyperbolicity to the rescue!

Global hyperbolicity to the rescue!

- M is globally hyperbolic.

Therefore causal diamonds in M are compact.

Global hyperbolicity to the rescue!

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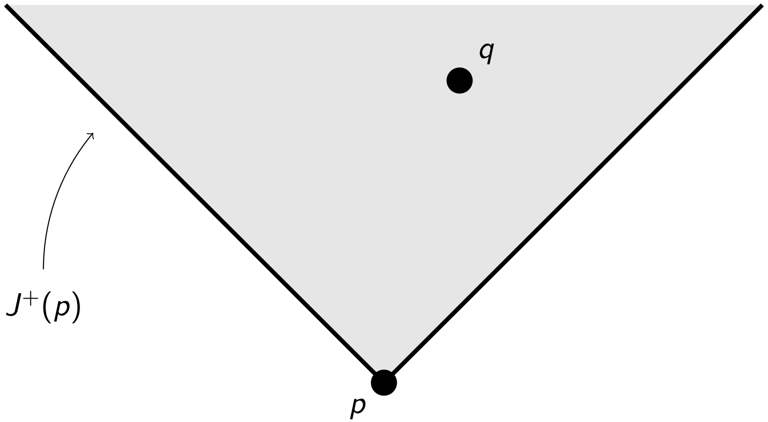
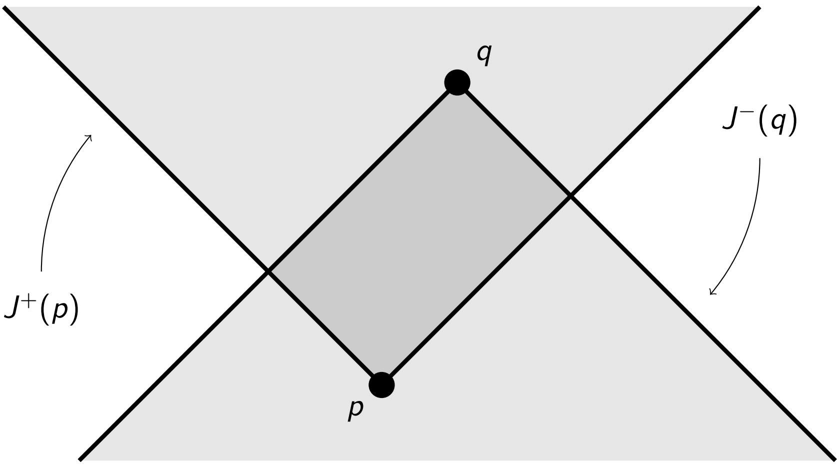
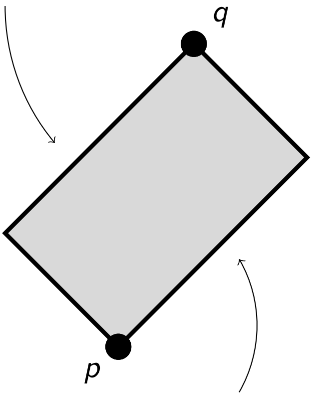


Global hyperbolicity to the rescue!



Global hyperbolicity to the rescue!

$J^+(p) \cap J^-(q)$



That's a compact set!

Narrow diamonds are subsets of M

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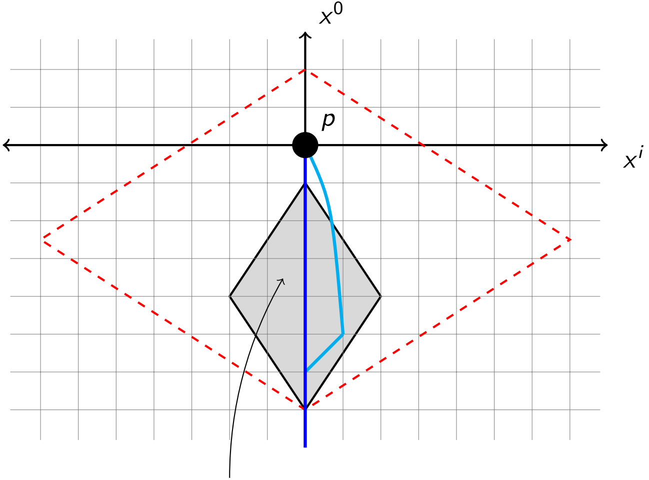

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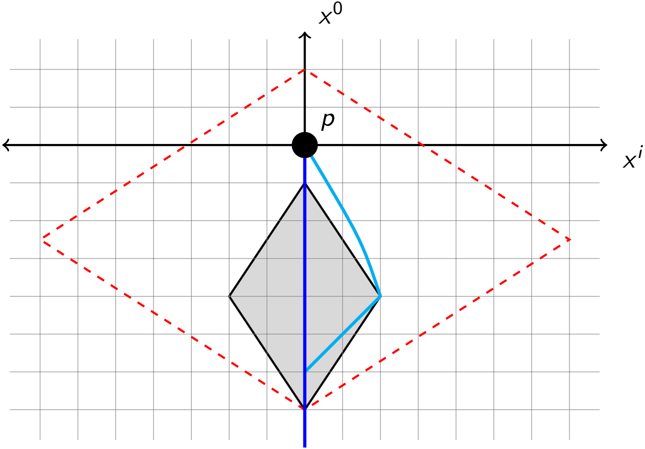
Narrow diamonds are subsets of M



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Thus...

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Theorem (Galloway, L., and Sbierski (2017))

A globally hyperbolic and timelike complete spacetime is C^0-inextendible.

Getting rid of global hyperbolicity

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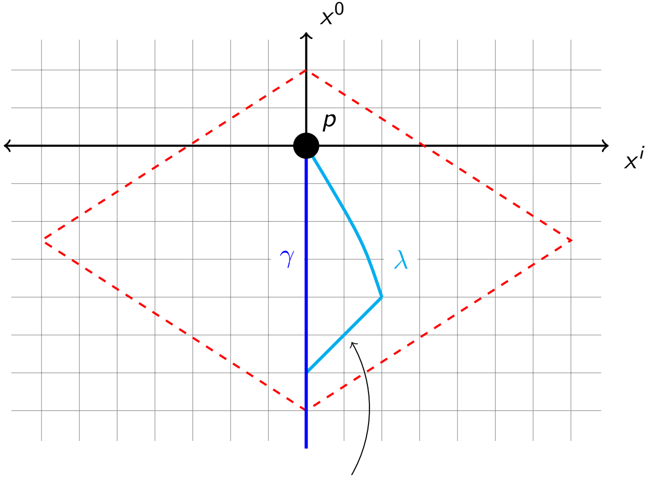
Theorem (Graf, L. (2017))

Causal maximizers in spacetimes with a Lipschitz metric are either timelike or null.

Getting rid of global hyperbolicity



This can't happen in a Lipschitz spacetime.

Thus...

Thus...

Theorem (Graf and L. (2017))

A timelike complete spacetime is inextendible as a spacetime with a Lipschitz metric (i.e. it's $C^{0,1}$-inextendible).

Thank you!

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Happy GregFest!