On the Geometry of Dark Matter

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The University of Miami

A Celebration of Mathematical Relativity
in honor of Greg Galloway

What is Dark Matter?

"A cosmic mystery of immense proportions, once seemingly on the verge of solution, has deepened and left astronomers and astrophysicists more baffled than ever. The crux... is that the vast majority of the mass of the universe seems to be missing."

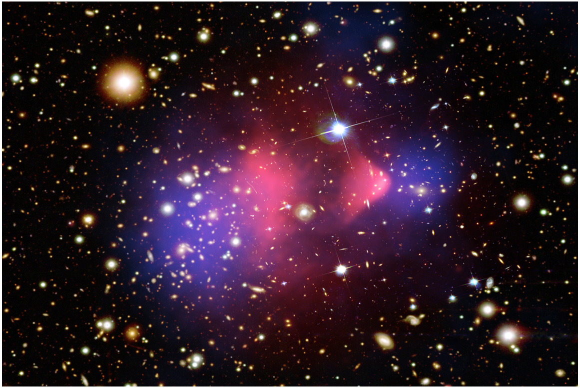
William J. Broad - 1984

"The discrepancy between what was expected and what has been observed has grown over the years, and we're straining harder and harder to fill the gap."

Jeremiah P. Ostriker - 1984

Dark Matter, as we understand, constitutes 27% of the mass of our universe, while visible matter constitutes around 5%.

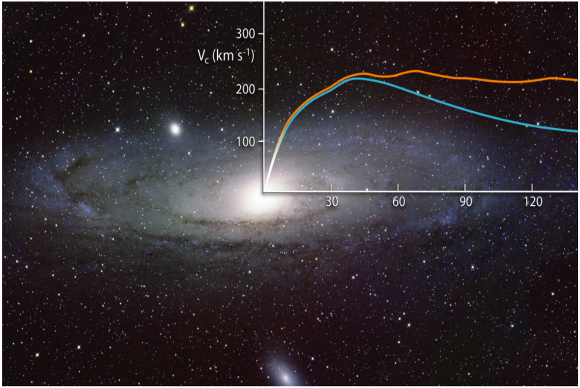
Data and Observation: Bullet Cluster



Dark matter is only known because of its gravity. It could also be called “large-scale unexplained curvature” of the universe.

The Bullet Cluster is an example where dark matter is separated from regular matter.

Data and Observation: Galactic Rotation Curves



Rotational velocity of spiral galaxies cannot be explained by only considering visible matter.

There is a discrepancy

$$V_{obs} > V_{visible}.$$

We can infer that galactic mass is over 90% dark matter.

What is dark matter?

- Particle physicist: It's a particle!

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Point: We all have our biases. Ultimately, the data will decide which models are the most descriptive.

What is dark matter?

- A particle. (WIMP)
- A wave. (WDM / SFDM / BEC / fuzzy DM / axions)
- A misunderstanding of gravity. (MoND)
- None of the above.

MoND (Modification of Newtonian Dynamics) predicts galactic rotation curves quite well, but is ruled out by observations like the Bullet Cluster.

WIMP (Weakly Interacting Massive Particles) theory is the most studied theory, is amazingly predictive on the scale of the universe, but is at odds with observations on the scale of galaxies. (Mass scale = 100 GeV)

WDM (Wave Dark Matter) makes very similar predictions, except on the scale of galaxies where its wave nature is most important. (boson mass scale $= 10^{-23} eV$)

Wave Dark Matter and Spiral Galaxies

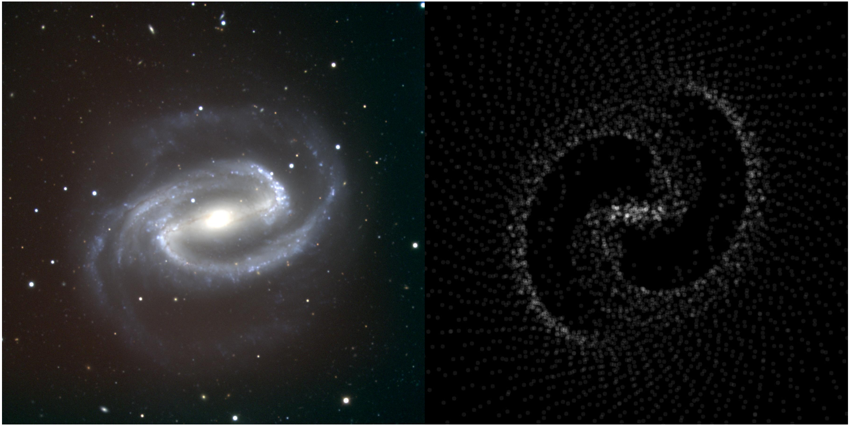
Idea 1: Natural geometric axioms motivate studying the Einstein-Klein-Gordon equations with a cosmological constant.

Idea 2: Wave types of equations, such as the Klein-Gordon equation, naturally form density waves in their matter densities.

Idea 3: Density waves in the wave equations, through gravity, naturally form spiral pattern density waves in the regular visible matter.

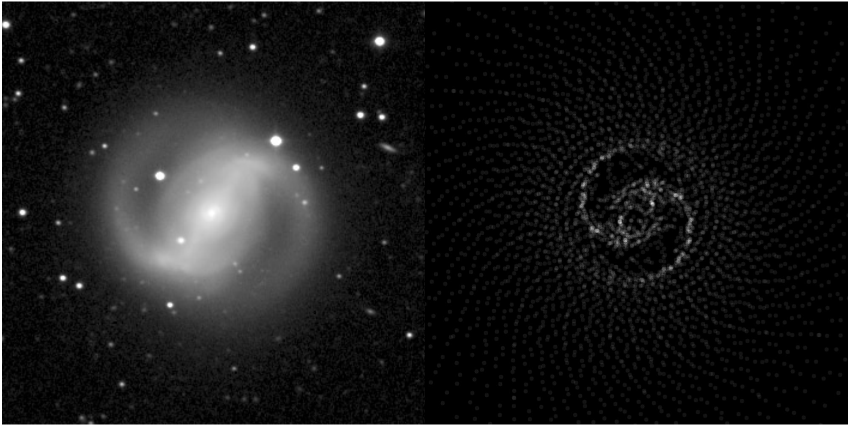
Comment: In the simulations below, the matter densities of the wave equation solutions result in rotating ellipsoidal gravitational potentials for the galaxies, which in turn cause the spiral patterns in the visible matter.

Spiral Galaxy Simulation #1



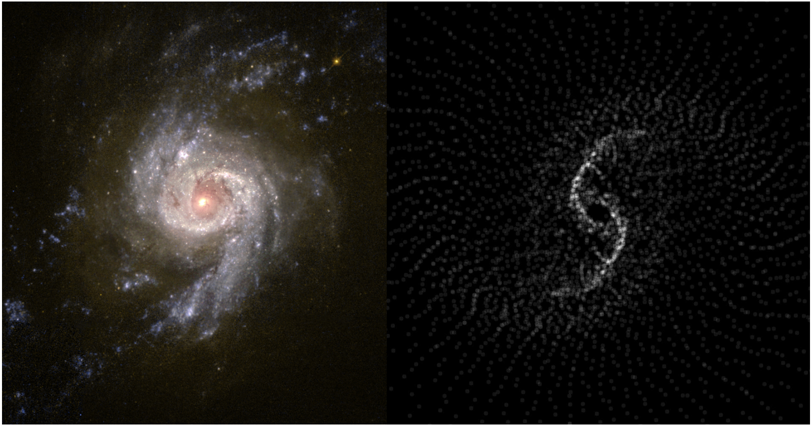
NGC1300 on the left, simulation on the right.

Spiral Galaxy Simulation #2



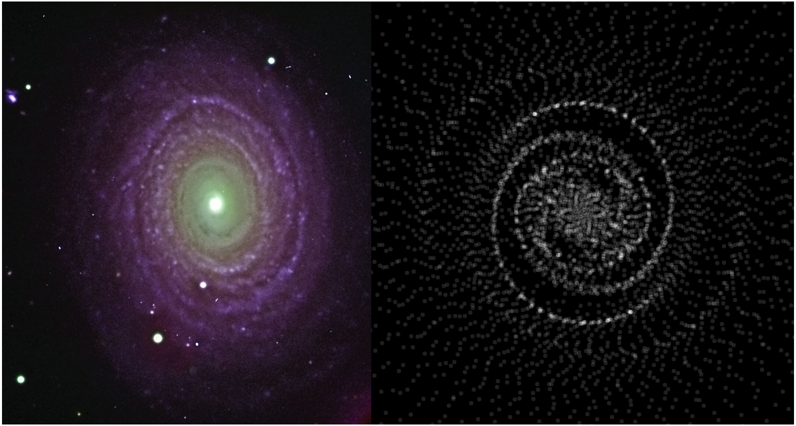
NGC4314 on the left, simulation on the right.

Spiral Galaxy Simulation #3



NGC3310 on the left, simulation on the right.

Spiral Galaxy Simulation #4



NGC488 on the left, simulation on the right.

What is the Action of the Universe?

The action of general relativity is

$$S(g) = \int (R_g - ...) \ dV_g \ .$$

For g to be a critical point of this action, we get

$$G = Ric_g - \frac{1}{2} R_g g = ...$$

which is usually written

$$G = 8\pi T.$$

What is the Action of the Universe?

The action of general relativity with a cosmological constant is

$$S(g) = \int (R_g - 2\Lambda - ...) \; dV_g \; .$$

For g to be a critical point of this action, we get

$$G + \Lambda g = ...$$

which is usually written

$$G + \Lambda g = 8\pi T.$$

What is the Action of the Universe?

The action of general relativity with a cosmological constant and a complex-valued scalar field is

$$S(g, f) = \int \left(R_g - 2\Lambda - 16\pi \left(\frac{|df|_g^2}{\Upsilon^2} + |f|^2 \right) - \ldots \right) \, dV_g \ .$$

What is the Action of the Universe?

The action of general relativity with a cosmological constant and a complex-valued scalar field is

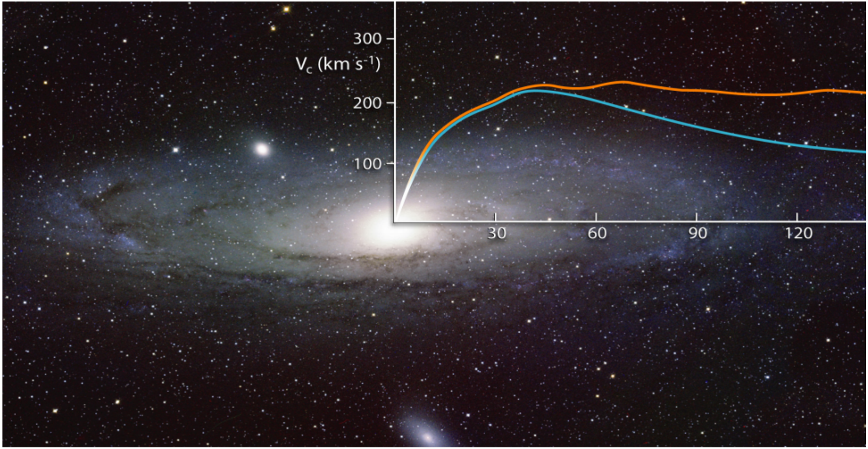
$$S(g, f) = \int \left(R_g - 2\Lambda - 16\pi \left(\frac{|df|_g^2}{\Upsilon^2} + |f|^2 \right) - \dots \right) dV_g \ .$$

For (g, f) to be a critical point of this action, we get

$$\begin{aligned} G + \Lambda g &= 8\pi \left(T + \frac{df \otimes d\bar{f} + d\bar{f} \otimes df}{\Upsilon^2} - \left(\frac{|df|_g^2}{\Upsilon^2} + |f|^2 \right) g \right) \\ D^2 f &= \Upsilon^2 f \end{aligned}$$

where D^2 is the wave operator equal to the divergence of the gradient with respect to the spacetime metric g.

Does WDM Predict Flat Rotation Curves?



Does WDM Predict Flat Rotation Curves?

Back of the envelope calculation in Minkowski: The Klein-Gordon equation

$$\Upsilon^2 f = D^2 f = \left(-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) f$$

has solutions

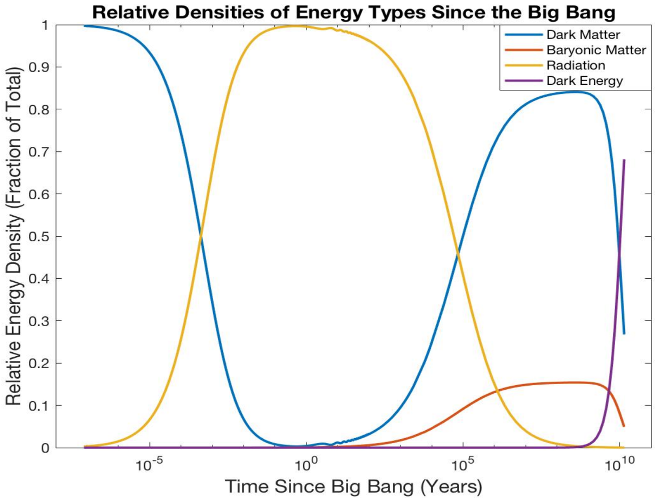
$$f(t,x) = a\, e^{i\omega t}\, \frac{\sin(kr)}{kr} \qquad\qquad (\omega^2 = k^2 + \Upsilon^2),$$

which leads to densities $\mu(r) \approx 2|f|^2 \approx b/r^2$, enclosed mass $m(r) = cr$, and circular speed as a function of radius

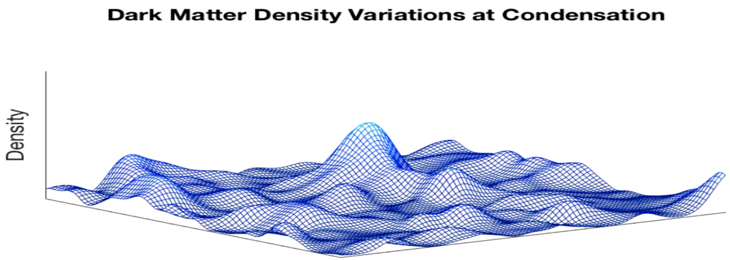
$$v^2 = ra = r\frac{m(r)}{r^2} = c,$$

which is a flat rotation curve. All circular orbits have the same speed. Answer: Yes!

A Brief History of Time



Cosmological Wave Dark Matter Solutions

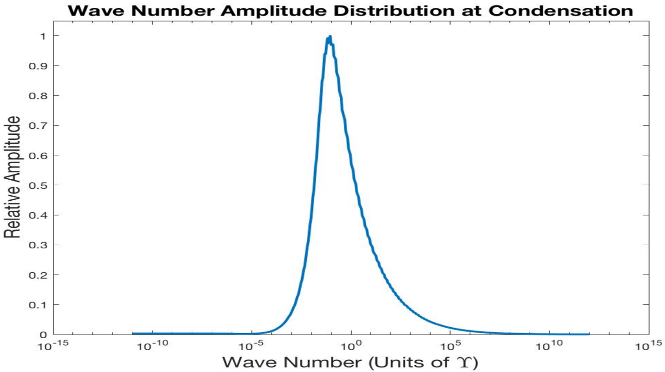


Fixing the spacetime metric as $-dt^2 + a(t)^2(dx^2 + dy^2 + dz^2)$,

$$f(t,x) = \sum_{j=1}^{N} A_j(t)\, e^{i\left(\vec{k}_j \cdot \vec{x} - s(t)\right)},$$

where $s(t)$ and $A_j(t)$ satisfy o.d.e.s depending on $|\vec{k}_j|$ and $a(t)$.

Amplitudes A as a function of $|\vec{k}|$



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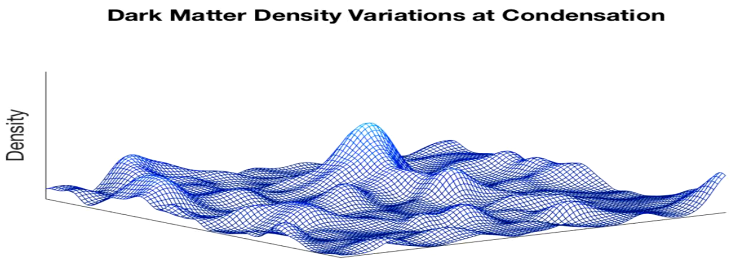
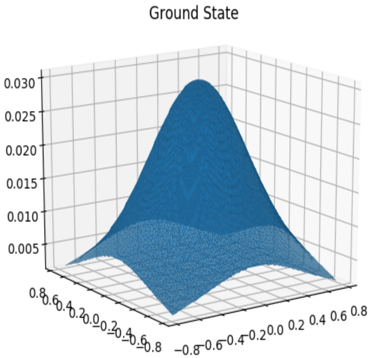
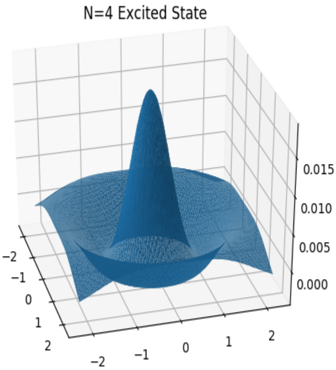


The group velocity of the wave peaks decreases as the universe expands. These wave peaks have gravity, which we believe eventually causes the infinitely expansive wave to break apart into droplets, when gravity finally overcomes the tendency of the waves to expand. We call this process condensation.

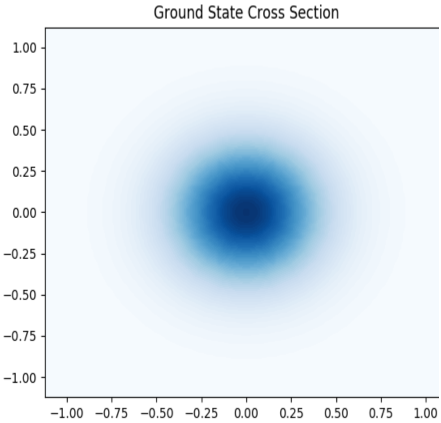
Spherically Symmetric Wave Dark Matter Solutions

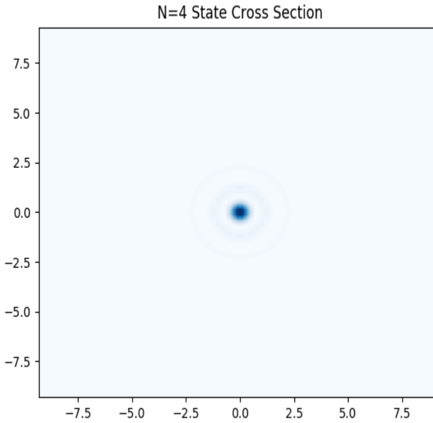




Graphs of $F(r)$, where $f(t,r) = F(r)e^{i\omega t}$, for some fixed ω.

Spherically Symmetric Wave Dark Matter Solutions





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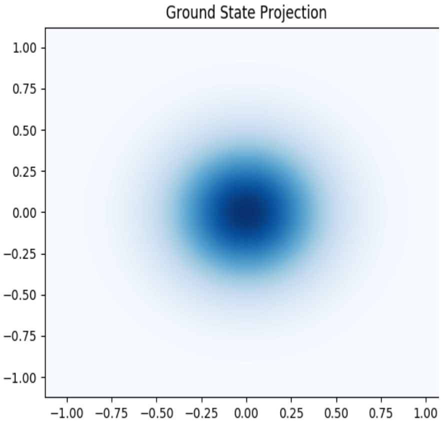

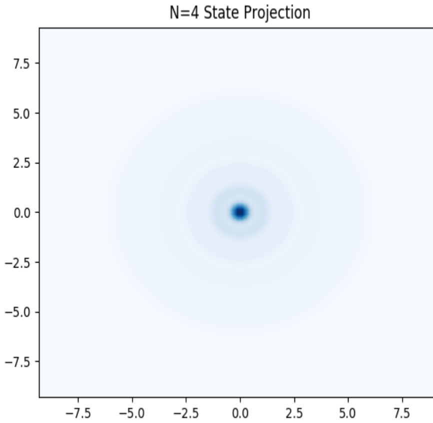

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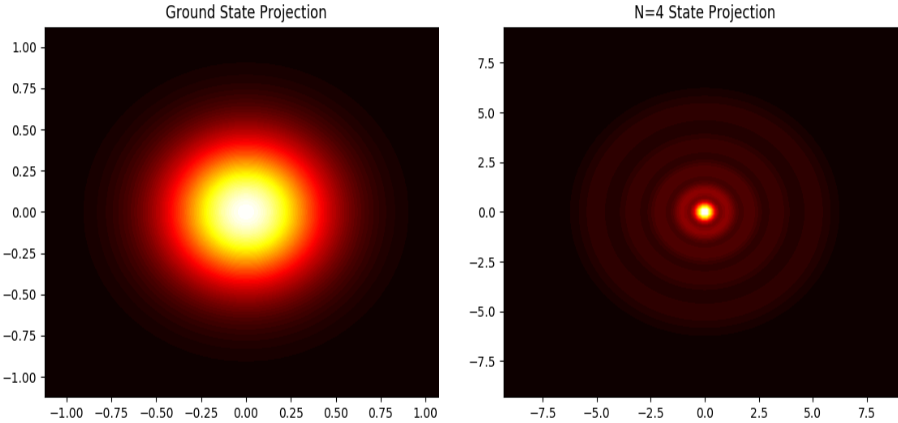


Spherically Sym. Static Solutions of Einstein-Klein-Gordon

If the spacetime metric and the scalar field have the form

$$\begin{aligned} g &= -e^{2V(r)}\, dt^2 + \Phi(r)^{-1} dr^2 + r^2 d\sigma^2_{S^2} \\ f(t,r) &= F(r)\, e^{i\omega t}, \qquad \Phi(r) = \left(1 - \frac{2M(r)}{r}\right) \end{aligned}$$

then the Einstein-Klein-Gordon equations are equivalent to

$$M_r = \frac{4\pi r^2}{\Upsilon^2}\left[(\Upsilon^2 + \omega^2 e^{-2V})F^2 + \Phi F_r^2\right]$$

$$\Phi V_r = \frac{M}{r^2} - \frac{4\pi r}{\Upsilon^2}\left[(\Upsilon^2 - \omega^2 e^{-2V})F^2 - \Phi F_r^2\right]$$

$$F_{rr} + \frac{2}{r}F_r + V_r F_r + \frac{\Phi_r}{2\Phi}F_r = \Phi^{-1}(\Upsilon^2 - \omega^2 e^{-2V})F$$

Boundary Conditions for Physical Solutions

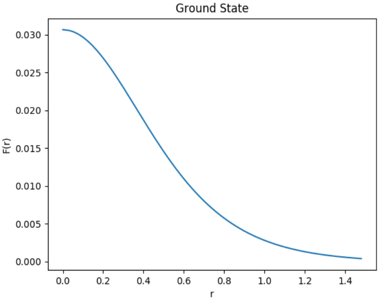
Physical Solutions to EKG must obey the following criteria:

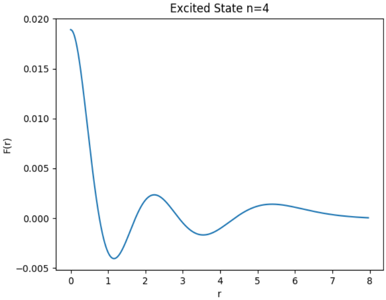
- $\lim_{r \to \infty} M(r) = M_\infty$ is finite.
- $\lim_{r \to \infty} F(r) = 0$.
- $M(0) = F_r(0) = V_r(0) = 0$.

Solving EKG only requires one to specify $F(0)$ and $V(0)$.

Ground States and Excited States: Graphs of $F(r)$

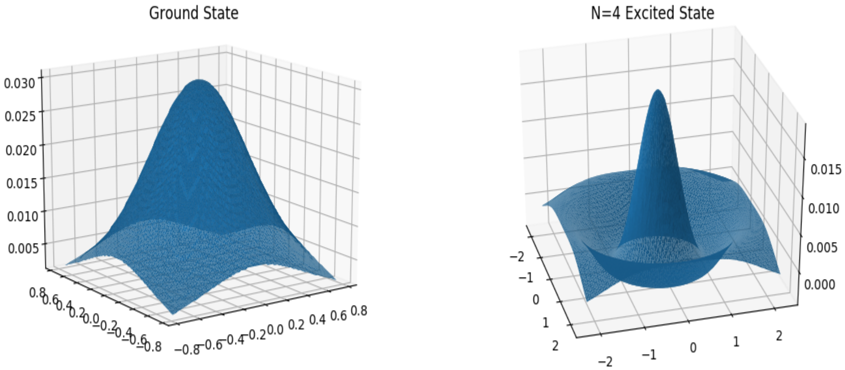
There are two types of bound, physical solutions





Ground States (F has no zeros) **Excited States** (F has n zeros)

Spherically Symmetric Wave Dark Matter Solutions



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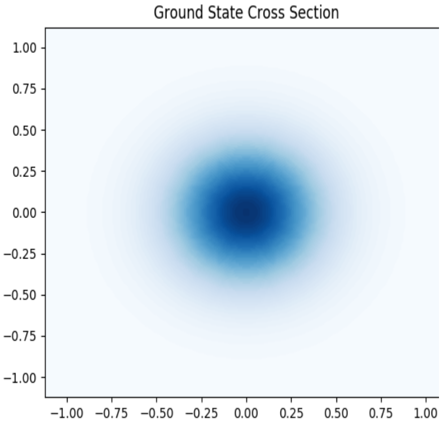
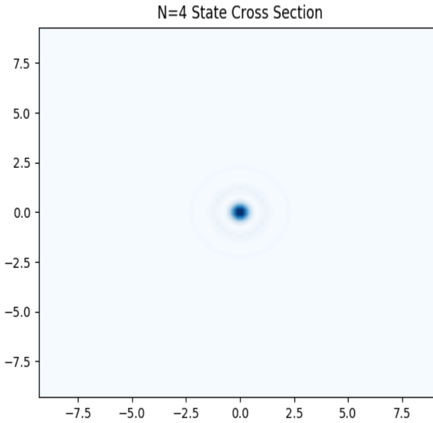
Spherically Symmetric Wave Dark Matter Solutions





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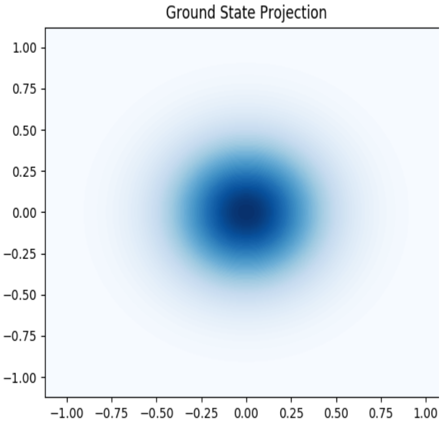
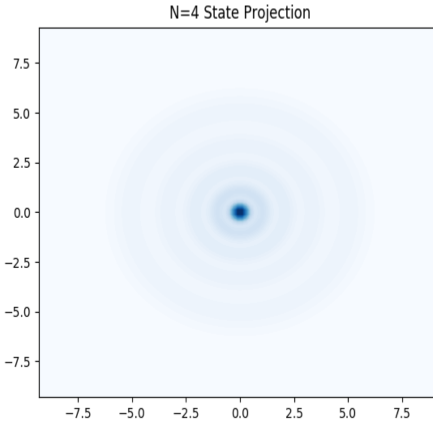
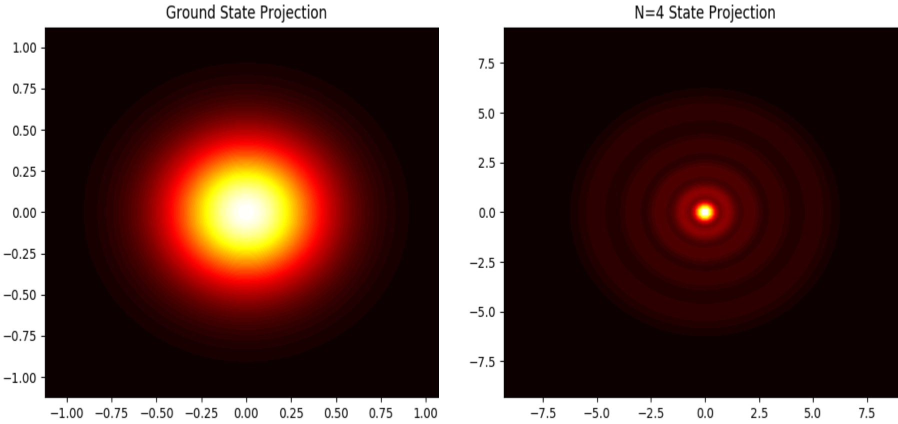
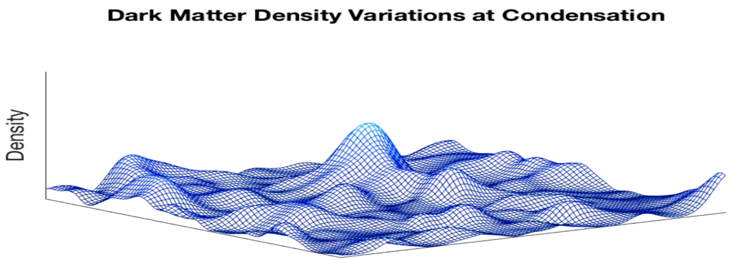




Spherically Symmetric Wave Dark Matter Solutions

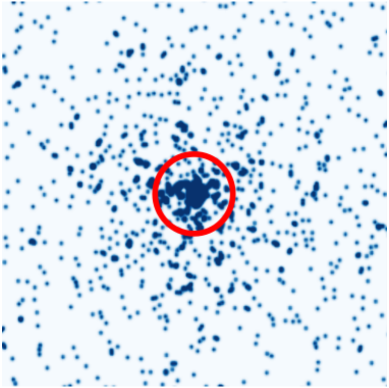


Cosmological Wave Dark Matter Solutions



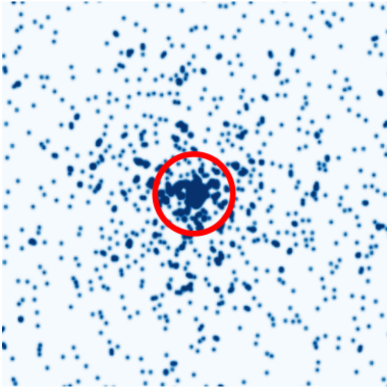
The group velocity of the wave peaks decreases as the universe expands. These wave peaks have gravity, which we believe eventually causes the infinitely expansive wave to break apart into droplets, when gravity finally overcomes the tendency of the waves to expand. We call this process condensation.

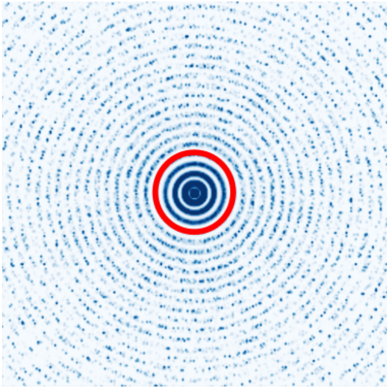
N-Body Problem with Many Ground States



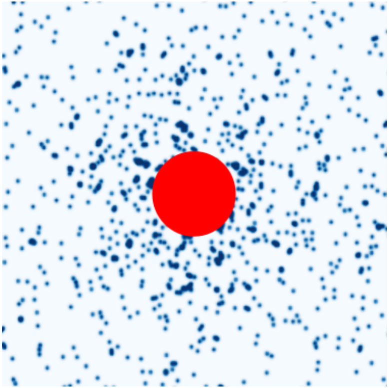
Left: Approximate N-body problem for a bunch of ground states.
Right: Globular cluster M80 / NGC 6093, a real-world solution to the N-body problem for several hundred thousand stars.

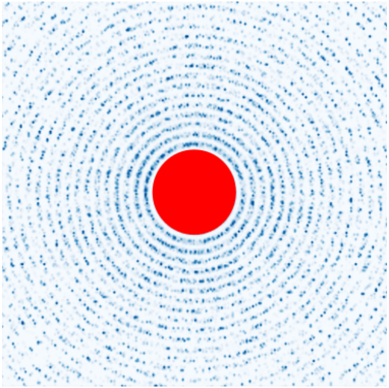
Approximation: Organize the N-Bodies into Rings





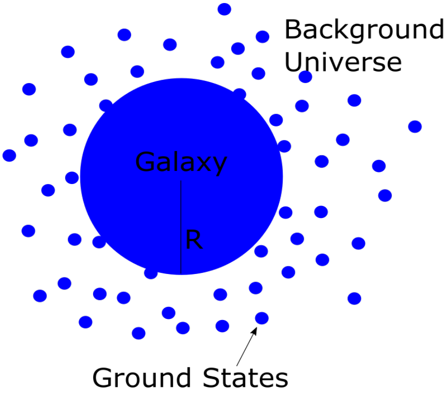
Solution to Nonlinear EKG Not Valid In Red Region





Goal: Replace red regions with fully nonlinear solutions to EKG. The problem on the right is a simplification of the problem on the left because it is very nearly spherically symmetric.

Background Boundary Conditions

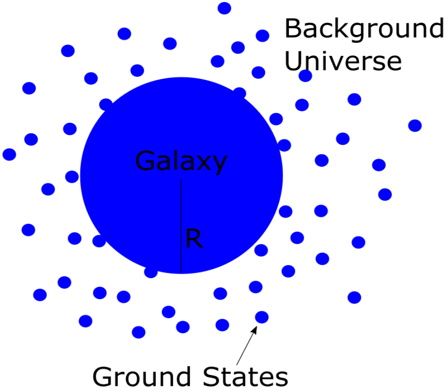


Hypothesis: The background universe contains ground states with central density amplitudes $\{A_{DM}\}$ and wavelengths $\{\lambda_{DM}\}$ whose distributions are roughly the same across the universe.

Boundary Conditions: At some radius R, characterizing the size of a galaxy, the spherically symmetric dark matter halo of the galaxy must satisfy:

- $A(R) = \overline{A_{DM}}$
- $\lambda(R) = \overline{\lambda_{DM}}$

Background Boundary Conditions

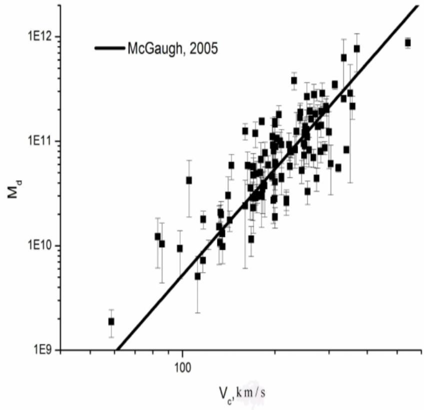


Boundary Conditions: At some radius R, characterizing the size of a galaxy, the spherically symmetric dark matter halo of the galaxy must satisfy:

- $A(R) = \overline{A_{DM}}$
- $\lambda(R) = \overline{\lambda_{DM}}$

Testable Prediction: For each value of R, the above boundary conditions give a unique spherically symmetric solution to the fully nonlinear EKG equations.

Testing WDM with the Tully-Fisher Relation



The Tully-Fisher Relation is an observed, linear trend which relates the rotational velocity of stars in spiral galaxies to their total mass.

When the total mass of a spiral galaxy is plotted against the characteristic velocity of the stars in that galaxy on a log-log plot, a line with slope 3 to 4 fits the data within error bars.

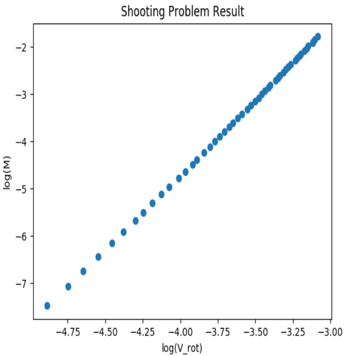
A One Parameter Family of Galaxies

For each value of $R > 0$:

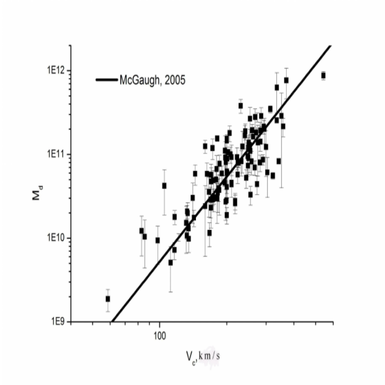
- Solve the boundary problem $A(R) = \overline{A_{DM}}$ and $\lambda(R) = \overline{\lambda_{DM}}$.
- Compute $M(R)$, the mass inside radiuis R.
- Compute the rotational velocity as $v_{rot}(R) = \sqrt{\frac{M(R)}{R}}$.
- Plot and fit $log(M(R))$ vs $log(v_{rot}(R))$.

To the extent that the population of ground states is the same everywhere in the universe, this take on WDM predicts a one parameter family of galaxies, approximately, with the parameter being the size R of the galaxies. Roughly speaking, this is what is observed.

Total Mass versus Speed of Stars in Spiral Galaxies



WDM predicted slope ≈ 3.2



Actual slope of observed galaxies is between 3 and 4.

Wave Dark Matter versus WIMP Dark Matter

- WIMP dark matter has been studied much more and has many successes, but has some issues on the scale of galaxies.
- Wave dark matter is believed to make nearly the same predictions, except on the scale of galaxies where its wave properties are important.

Issue #1: For WIMPs, N-body problem simulations limit to the NFW (Navarro-Frenk-White) profile, where the dark matter density goes to infinity at the center of galaxies. So far, this is not observed, and may be contrary to the data.

Issue #2: For WIMPs, there is no reason to get a lower bound on the masses of galaxies. However, observationally there is a lower bound on the masses of galaxies. The smallest galaxies are called dwarf spheriodal galaxies, which are roughly spherically symmetric.

Virtues of Wave Dark Matter

- Waves spread, so the dark matter density at the centers of galaxies is predicted to be bounded, as is thought to be observed.
- Ground states give a lower bound on the masses of galaxies.
- Roughly flat rotation curves, as observed.
- Provides a mechanism for spiral patterns in galaxies.
- Like WIMPS, is a form of cold dark matter, known to model the universe extremely well, on scales larger than galaxies.
- Predicts a 1 parameter family of galaxies (roughly), consistent with observations.
- Fits the Tully-Fisher data with slope 3.2, assuming the boundary conditions described in this presentation.

Virtues of Wave Dark Matter

Wave dark matter also has a beautiful geometric motivation and a relatively simple geometric action, considering it may describe 95% of the mass of the universe:

$$S(g, f) = \int \left(R_g - 2\Lambda - 16\pi \left(\frac{|df|_g^2}{\Upsilon^2} + |f|^2 \right) - \ldots \right) \, dV_g \ .$$

Alas, that doesn't count as evidence, but it does inspire us to continue the vigorous pursuit of understanding how geometric analysis describes the universe in which we live in surprising and remarkable ways.

Thank you!

"For me, it is far better to grasp the universe as it really is than to persist in delusion, however satisfying and reassuring." - Carl Sagan