1. Let $V$ be a vector space. Prove that the list $(v_1, \ldots, v_n)$ is a basis for $V$ if and only if each $v_i \neq 0$ and $V = \langle v_1 \rangle \oplus \ldots \oplus \langle v_n \rangle$.

2. Let $V$ be a finite dimensional vector space and let $W$ be a subspace. Prove that $V \cong W \oplus V/W$.

3. Let $V$ be a finite dimensional vector space and let $\hat{V}$ be the dual space. Prove that if $v_1 \neq v_2$ in $V$ then there is an $f \in \hat{V}$ such that $f(v_1) \neq f(v_2)$.

4. Let $V$ be a vector space over the field $F$. Let $T : V \to V$ be a linear map. Let $V_T$ be the $F[x]$-module on the set $V$ with same addition as $V$ and with scalar multiplication given by $f(x)v := f(T)(v)$. Show $V_T$ is an $F[x]$-module.

5. Let $A$ and $B$ be submodules of $R$-module $M$. Prove
   
   (a) $A \cap B$ is a submodule of $M$.
   (b) $A + B$ is a submodule of $M$
   (c) $(A + B)/B$ is isomorphic to $A/(A \cap B)$.

6. Let $M$ be an $R$-module and let $U_1, \ldots, U_n$ be submodules. Prove $M = U_1 \oplus \ldots \oplus U_k$ and only if $M = U_1 + \ldots + U_k$ and $U_i \cap (U_1 + \ldots + U_{i-1}) = \{0\}$ for all $i \geq 2$.

7. Let $R = \{ f(x) \in \mathbb{Z}[x] : f(x) = a_0 + a_2x^2 + \ldots + a_nx^n \}$.
   
   (a) Show $\mathbb{Z}[x]$ is an $R$-module.
   (b) Show $\{1, x\}$ is a minimal generating set.
   (c) Show the $R$-module $\mathbb{Z}[x]$ does not decompose into a direct sum of cyclic submodules.

8. An $R$-module is said to be irreducible if its only submodules are $\{0\}$ and $M$. Suppose that $R$ has unity and $M$ is a unital irreducible $R$-module. Show that $M$ is cyclic.