(1) Given a commutative ring $R$, the group of units of $R$ is $U(R) = \{\text{all units in } R\}$. Prove that $U(R)$ is a multiplicative group.

(2) If $X$ is a set, prove the Boolean group $\mathcal{B}(X)$ under addition defined by symmetric difference, is a commutative ring with multiplication defined by $UV = U \cap V$. Hint: Recall standard facts from set theory such as De Morgan’s law.

(3) Let $\mathbb{F}_2 = \{0, 1\}$, and define addition and multiplication by $0+0 = 1+1 = 0$, $0+1 = 1+0 = 1$, $0 \cdot 0 = 0 \cdot 1 = 1 \cdot 0 = 0$, and $1 \cdot 1 = 1$. Explain why $\mathbb{F}_2$ is a field. (You can assume that we have the associative and distributive properties).

(4) Let $p$ be a prime number. The ring of integers modulo $p$ is a field with $p$ elements, denoted $\mathbb{F}_p$.
   (a) Explain why $\mathbb{F}_p - \{0\}$ is a group under multiplication.
   (b) Use Lagrange’s Theorem to show that $a^{p-1} = 1$ for all $a \in \mathbb{F}_p - \{0\}$.
   (c) Prove that $a^p = a$ for all $a \in \mathbb{F}_p$. Hint: treat cases $a = 0$ and $a \neq 0$ separately.
   (d) Find a nonzero polynomial in $\mathbb{F}_p[x]$ which vanishes at every point of $\mathbb{F}_p$. Hint: use part c.

(5) Let $F$ be a finite field with $q$ elements. Adapt the argument of the previous exercise to prove that $x^q - x$ is a nonzero polynomial in $F[x]$ which vanishes at every point of $F$. 